Information Theory and Networks
Lecture 15: Stream Coding

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Part I
Stream Coding

Dr. Egon Spengler: There's something very important I forgot to tell you.
Dr. Peter Venkman: What?
Dr. Egon Spengler: Don't cross the streams.
Dr. Peter Venkman: Why?
Dr. Egon Spengler: It would be bad.

[some time later]

Dr. Egon Spengler: [hesitates] We'll cross the streams.
Dr. Peter Venkman: 'Scuse me Egon? You said crossing the streams was bad!
⋮
Dr. Egon Spengler: Not necessarily. There's definitely a very slim chance we'll survive.

Ghost Busters

Section 1
Problems and Solutions

Problem of Block Huffman Coding

- Huffman coding needs probabilities
  - do you estimate them from the file – two passes?
  - or use generic probabilities – not quite accurate for a particular file?
- Blocks of length n have d^n possible "symbols"
  - estimating small probabilities is hard
  - do you include the (large) dictionary in the compressed file?
  - Huffman needs complete recalculation to change the block size

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Stream Coding

- **Block Coding:**
  - fixed set of input symbols (fixed block size)
  - same (possibly variable length) codes used through one file or transmission
  - transmitter and receiver need the same code dictionary
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  - process file (or transmission) as it comes
  - code dictionary adapts as it goes
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Section 2

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Arithmetic Coding

- RLE (see last lecture)
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Lempel-Ziv-Welch (LZW) [ZL78, Wel84]

- Simple version encodes series of 8-bit data (e.g., ASCII)
- 12 bit "codewords"
  - codes from 0-255 represents an 8-bit character (directly)
  - codes from 256-4095 refer to a dictionary, based on the data
- goal – replace long, repeated strings with a simple code (number)
  - construct the dictionary of strings as you go
  - as the file is processed, we get better and better compression (we hope)
- encoding
  - dictionary starts with all strings of length 1
  - repeat
    - find longest string W in dictionary that matches current input
    - put dictionary index for W in output, and remove W from input
    - add W followed by next symbol in the input to the dictionary
- decoding
  - iteratively translate and build the dictionary
  - don't need to transmit the dictionary

The GIF image format (1987) uses Lempel-Ziv-Welch.
An old utility `compress` also used it.

Arithmetic Coding

- Use an adaptive Bayesian model for the probabilities
  - estimate it as we go along
- Encode with a Shannon-Fano-Elias-like code
- Decoder decodes symbols, and uses the same method to estimate probabilities, and hence derive the codes as you go along.

Bayesian Model

- Take source alphabet $\Omega = \{a_1, a_2, \ldots, a_l\}$ where a_l indicates "end of transmission"
- Source produces $X_1, X_2, \ldots \in \Omega$
- Both source coder, and receiver build a predictive probability distribution
$$p(X_n = a_i | X_{n-1}, X_{n-2}, \ldots, X_1)$$
- For example, use Bayesian estimates
  - fix probability of $a_l = 0.15$
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We don't, in general, assume that the X_i are IID.

We must fix $a_l = 0.15$ because you won't see that symbol until the end.

Shannon-Fano-Elias (SFE) Coding

- Lets do it for IID symbols:
$$p(X_n = a_i | X_{n-1}, X_{n-2}, \ldots, X_1) = p(X_n = a_i)$$
- Construct the CDF $F(x) = P(X \leq x)$
$$F(x) = \sum_{a<x} p(a)$$
and from this a new function
$$\bar{F}(x) = \sum_{a<x} p(a) + p(x)$$
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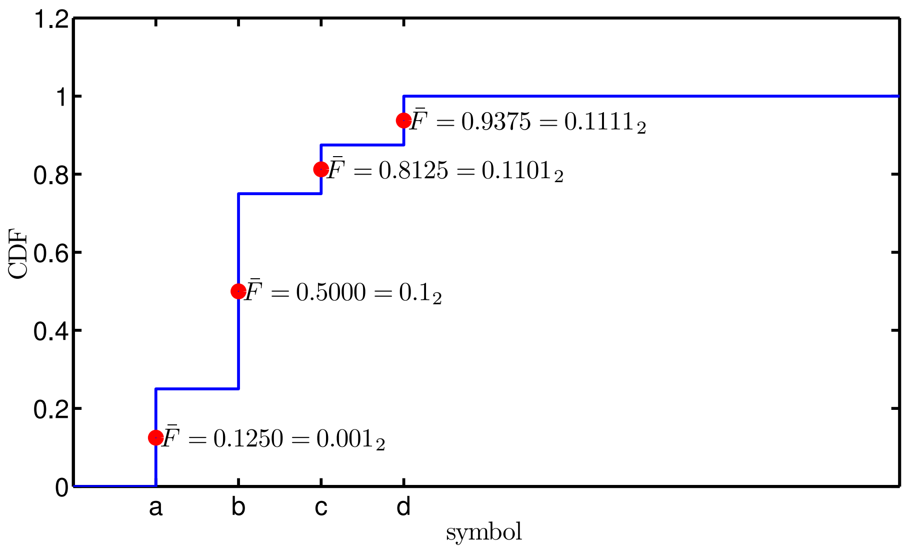
SFE Illustration 1

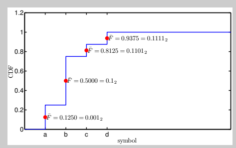
X	$p(x)$	$F(x)$	$\bar{F}(x)$	$\bar{F}(x)$ in binary	$\ell(x)$	codeword
a	0.25	0.25	0.125	0.001_2		
b	0.5	0.75	0.5	0.1_2		
c	0.125	0.875	0.8125	0.1101_2		
d	0.125	1.0	0.9375	0.1111_2		
$H(X)$	1.75					
$E_p\ell$						

[CT91, Example 5.9.1, p.103].

Note that even though the code is finite here, it wouldn't be prefix free if we just used the binary for $\bar{F}$.

SFE Illustration 1





SFE Coding

- Lets do it for IID symbols:

$$p(X_n = a_i | X_{n-1}, X_{n-2}, \ldots, X_1) = p(X_n = a_i)$$

  - Optimal codeword length approximation

$$\ell(a_i) = \left\lceil \log\left(\frac{1}{p(a_i)}\right) \right\rceil$$

  - Have one extra bit (see why in a second)

$$\ell(a_i) = \left\lceil \log\left(\frac{1}{p(a_i)}\right) \right\rceil + 1$$

  - Truncate the codes from $\bar{F}$ to this length

SFE Illustration 1

X	$p(x)$	$F(x)$	$\bar{F}(x)$	$\bar{F}(x)$ in binary	$\ell(x)$	codeword
a	0.25	0.25	0.125	0.001_2	3	001
b	0.5	0.75	0.5	0.1_2	2	10
c	0.125	0.875	0.8125	0.1101_2	4	1101
d	0.125	1.0	0.9375	0.1111_2	4	1111
$H(X)$	1.75					
$E_p \ell$						2.75 bits per symbol

NB: Huffman code for this case achieves the entropy bound. Last bit of the last two could be omitted, but we can't just drop a bit from all of them or it isn't prefix free.

SFE Coding Theory

What are we trying to achieve?

- Code's related to $\bar{F}$ (we'll see why later)
- Optimal(ish) (code lengths given by $\simeq \ell(x)$ above)
- Prefix free (hence need for extra bit)

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- Code's related to $\bar{F}$ (we'll see why later)
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SFE Coding Theory

Use the first $\ell(x)$ bits of $\bar{F}(x)$

- possible error in rounding off
$$\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \frac{1}{2^{\ell(x)}}$$
- take
$$\ell(x) = \left\lceil \log\left(\frac{1}{p(x)}\right) \right\rceil + 1$$
- then
$$\frac{1}{2^{\ell(x)}} < \frac{p(x)}{2} = \bar{F}(x) - F(x-1)$$
- thus
$$\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \bar{F}(x) - F(x-1)$$
- thus the $\ell(x)$ length code is in the interval we want it to be in

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- take $\ell(x) = \left\lceil \log\left(\frac{1}{p(x)}\right) \right\rceil + 1$
- then $\frac{1}{2^{\ell(x)}} < \frac{p(x)}{2} = \bar{F}(x) - F(x-1)$
- thus $\bar{F}(x) - \lfloor \bar{F}(x) \rfloor_{\ell(x)} < \bar{F}(x) - F(x-1)$
- thus the $\ell(x)$ length code is in the interval we want it to be in

Denote a number x rounded down (by say a floor function) to m digits by
$$\lfloor x \rfloor_m$$
Note also that
$$\frac{p(x)}{2} = \bar{F}(x) - F(x-1)$$
by definition of $\bar{F}$.

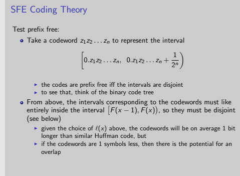
SFE Coding Theory

Test prefix free:

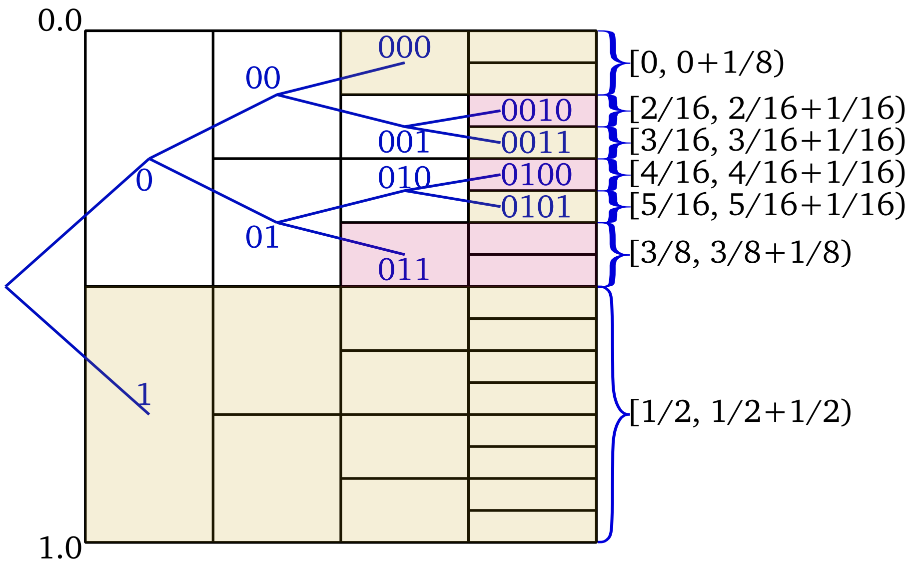
- Take a codeword $z_1 z_2 \dots z_n$ to represent the interval

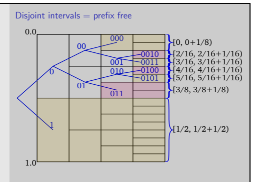
$$\left[0.z_1 z_2 \dots z_n, \;\; 0.z_1 z_2 \dots z_n + \frac{1}{2^n}\right)$$

  - the codes are prefix free iff the intervals are disjoint
  - to see that, think of the binary code tree
- From above, the intervals corresponding to the codewords must like entirely inside the interval $\big[F(x-1), F(x)\big)$, so they must be disjoint (see below)
  - given the choice of $\ell(x)$ above, the codewords will be on average 1 bit longer than similar Huffman code, but
  - if the codewords are 1 symbols less, then there is the potential for an overlap



Disjoint intervals = prefix free





Assume codeword $z_1 z_2 \dots z_n$ is equivalent the interval

$$\left[0.z_1 z_2 \dots z_n, \;\; 0.z_1 z_2 \dots z_n + \frac{1}{2^n}\right)$$

We can see immediately that prefix free codes are equivalent to non-overlapping intervals.
Colours are just there so we can distinguish adjacent intervals.

SFE Illustration 2

X	$p(x)$	$F(x)$	$\bar{F}(x)$	$\bar{F}(x)$ in binary	$\ell(x)$	codeword
a	0.25	0.25	0.125	0.001_2	3	001
b	0.25	0.5	0.375	0.011_2	3	011
c	0.2	0.7	0.6	$0.1\overline{0011}_2$	4	1001
d	0.15	0.85	0.775	$0.110\overline{0011}_2$	4	1100
e	0.15	1.0	0.925	$0.111\overline{0110}_2$	4	1110
$H(X)$	2.2855					
$E_p\ell$						3.5 bits per symbol

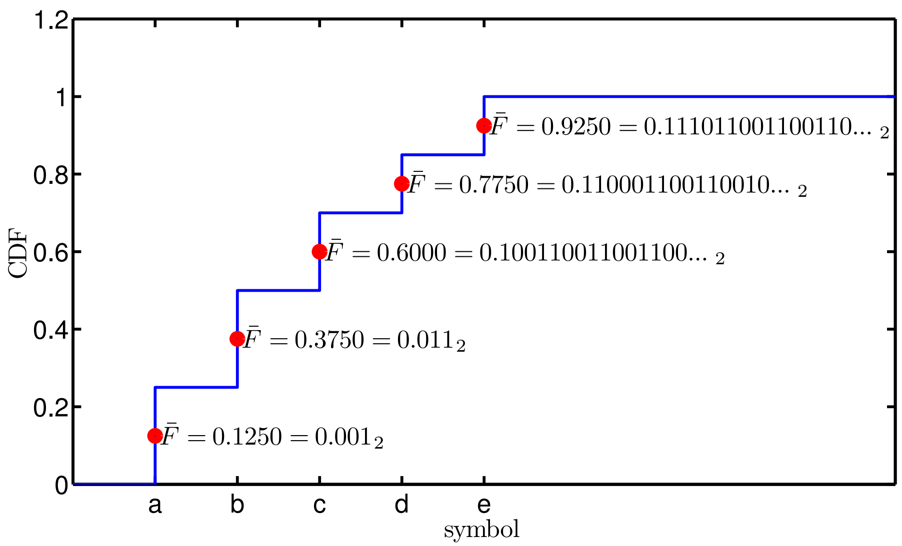
[CT91, Example 5.9.2, p.103].

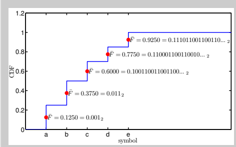
We denote repeated digits using, e.g.,

$$0.1\overline{0011}_2 = 0.1001100110011\ldots$$

NB: average code length is about 1.2 bits longer than entropy bound.

SFE Illustration 2





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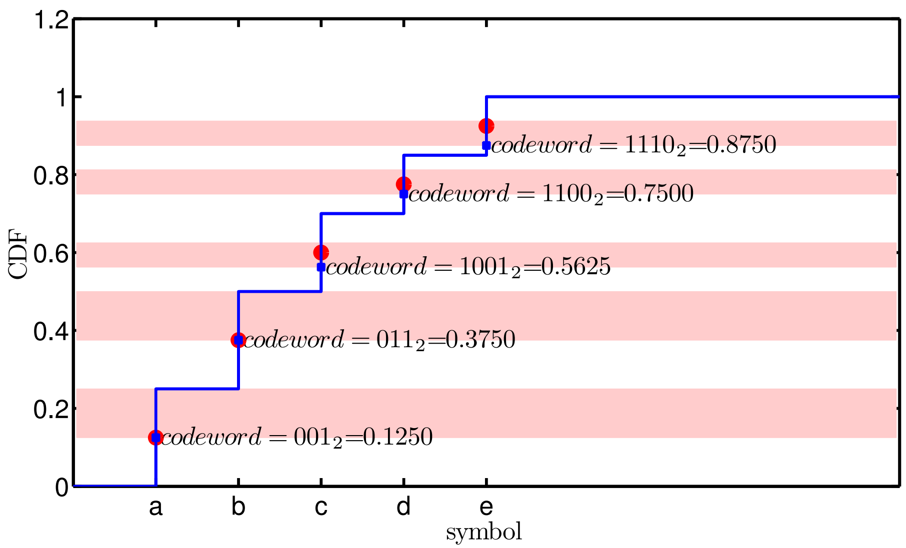

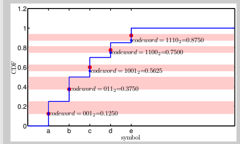

Arithmetic Coding

The Coding Step

- We can see the SFE coding isn't the most efficient, but it has the huge advantage that we can build hierarchical codes in a similar way.
- Assume we can estimate

$$p(X_n|X_{n-1}, X_{n-2}, \ldots, X_1)$$

- Imagine we could construct the SFE code for this
  - if the first bit of that code would result in an interval that is entirely inside the step, then we can use it
  - if not, keep that bit in mind, and then divide the current step into components according to the next probability distribution
  - eventually, we can start fixing some of the old bits
- Iteratively perform these operations, along with probability estimation.

[Mac11, 6.2,pp.111-118] and [CT91, 5.10,p.104-107] have more details.

Others

- DEFLATE: png (images), gzip and zip
- Apple Lossless (ALAC - Apple Lossless Audio Codec)
- Free Lossless Audio Codec (FLAC)
- WMA Lossless (Windows Media Lossless)

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Further reading I

- Thomas M. Cover and Joy A. Thomas, *Elements of information theory*, John Wiley and Sons, 1991.
- David J. MacKay, *Information theory, inference, and learning algorithms*, Cambridge University Press, 2011.
- Terry Welch, *A technique for high-performance data compression*, IEEE Computer (1984), 819, http://ieeexplore.ieee.org/xpls/abs_all.jsp?tp=&arnumber=1659158&isnumber=34743&tag=1.
- Jacob Ziv and Abraham Lempel, *Compression of individual sequences via variable-rate coding*, IEEE Transactions on Information Theory (1978).