# Complex-Network Modelling and Inference

## Lecture 12: Random Graphs: spatially-embedded and small-world networks


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## Section 1

# Spatially-Embedded Random Graphs

# Space and networks

Many networks have nodes embedded in space

- Most physical networks
  - *e.g.*, electricity
  - *e.g.*, Internet
  - *e.g.*, air-plane routes
- Many biological networks
  - *e.g.*, animal interaction networks
  - *e.g.*, epidemiological contact graphs
  - *e.g.*, neural networks
- Many social networks
  - most people have a locus around which they spend most time

# Space and networks

In many settings a longer link is more expensive in some sense

- *e.g.*, Longer electricity wires or telephone cables are more costly to build
- *e.g.*, Contact between animals requires them to move over larger distances, and hence expend more energy.
- *e.g.*, Neural networks – sending signals over longer distances can be slower, costing speed.
- *e.g.*, Wireless connections require more power over longer distances, and will therefore create more interference.

The natural consequence is that longer links will be less likely in spatially embedded networks.

# Waxman graphs

Early examples

- [Wax88] Take $n = |N|$ nodes, distributed uniformly at random on a square (or potentially some other space)
- [ZCB96] connect nodes $i$ and $j$ randomly with probability

$$p = \alpha f(d_{ij}).$$

  where $d_{ij}$ is the Euclidean distance between them.
  - example 1
    $$p = \alpha\, e^{-\beta d_{ij}}.$$
  - example 2
    $$p = \alpha\, u(d_{ij} - r),$$
    where $u(x)$ is the unit step function (at zero).
- key point is that links still chosen independently *conditional* on the locations of the nodes

# Spatially Embedded Random Graphs (SERNs)

1. Place $n$ nodes *randomly* within a defined region $R$ of a metric space $\Omega$
    1. randomly usually means uniformly, *i.e.*, the probability of a node occurring in any sub-region $r$ is proportional to the area of $r$.
    2. a metric space means it has a distance metric $d(x, y)$
2. Probability of an edge $(i, j)$ is

$$p_{ij} = q\, f_\theta(s\, d_{i,j}),$$

where

$$\begin{aligned} f(\cdot) &= \text{a distance deterrence function,} \\ q &= \text{a thinning parameter,} \\ s &= \text{a scale parameter,} \\ \theta &= \text{other parameters.} \end{aligned}$$

Don't panic! Most cases use standard Euclidean spaces, and simple deterrence functions.

# *e.g.*, Waxman Random Graphs

Points are chosen in the unit square, and

$$p(d_{i,j}) = q\, e^{-s d_{i,j}}.$$

Notes

- I use a different parameterisation from the literature
  - to be consistent with other models
  - because the literature gets it wrong about half the time
- The idea has been reinvented multiple times, but Waxman was the first as far as I know.

# *e.g.*, Random Geometric Graphs

Points are chosen in the unit square, and

$$p(d_{i,j}) = q\, H(1 - s d_{i,j}),$$

where $H(\cdot)$ is the Heavyside step function.

Notes

- Also called
  - random plane network [Gil61]
  - random connection models
  - random distance graphs

# Other common examples

- GER: $p_{i,j} = q$, where $q \in (0,1]$ [Gil59, ER60];
- Clipped Waxman: $p_{i,j} = \min\left(q\, e^{-sd_{i,j}}, 1\right)$, where $s \in [0,\infty), q \in (0,\infty)$;
- Mixed Waxman-threshold: $p_{i,j} = q\, e^{-sd_{i,j}} H(r - sd_{i,j})$, where $s \in [0,\infty), q \in (0,1], r \in [0,\infty)$;
- Power law: $p_{i,j} = q\,(1 + s\, d_{i,j})^{-\theta_2}$, (*e.g., range-dependent random graphs*) [Gri02, Far02, GD07, HM03], and
- Cauchy: $p_{i,j} = q\,\left(1 + (sd_{i,j})^2\right)^{-1}$ [Avi08].
- Exponential: $p_{i,j} = \dfrac{q\, e^{-d_{i,j}}}{L - d_{i,j}}$, [ZCD97];

# Common features

- Distance deterrence function $f(\cdot)$ is non-increasing
  - it doesn't have to be, but all cases I know have this feature, and it relates to the motivation
- Usually in a finite region, but don't have to be, and often we are interested in asymptotic limits
- Underlying point process of node locations is a n-D Poisson Process
  - doesn't have to be, but not much work where it isn't
  - produces the uniformly at random result

# Sidebar on the line picking problem

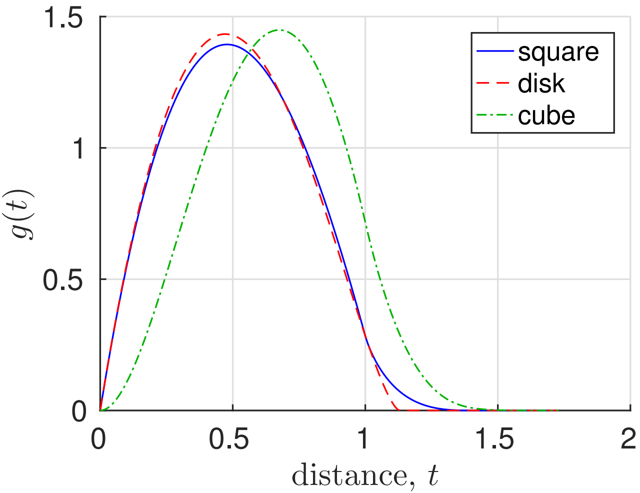
The *line-picking problem* is the problem of finding the probability distribution of the length of a random line in some region. More precisely, choose two points at random in the region, what is the distribution of distances between them.

- Lots of results are known
  - *e.g.*, square line picking, the probability density is

$$g(t) = \begin{cases} 2t(t^2 - 4t + \pi) \ \text{ for } 0 \leq t \leq 1, \\ 2t\left[4\sqrt{t^2-1} - (t^2 + 2 - \pi) - 4\tan^{-1}\left(\sqrt{t^2-1}\right)\right] \\ \qquad \text{for } 1 \leq t \leq \sqrt{2}. \end{cases}$$

- It's not hard to calculate numerically even when it isn't known
- Major determining factors:
  - size of region
  - dimension of space
  - "aspect ratio" of region

# Sidebar on the line picking problem



# Waxman in detail

Points chosen in $R \subset \Omega$, and

$$p(d_{i,j}) = q\, e^{-sd_{i,j}}.$$

Probability of an arbitrary link (prior to knowing the distances):

$$Prob\{(i,j) \in \mathcal{E} \mid q, s\} = q \int_0^\infty \exp(-st) g(t)\, dt = q\tilde{G}(s),$$

where $\tilde{G}(s)$ is the Laplace transform of $g(t)$ (from line-picking)
When

- $s = 0$, the Laplace trans. becomes the normalisation constraint so $p(d_{i,j}) = q = Prob\{(i,j) \in \mathcal{E} \mid q, s\}$

# Waxman in detail

Average number of edges

$$\mathbb{E}[|E|] = \frac{n(n-1)}{2} Prob\{(i,j) \in \mathcal{E} \mid q, s\} = \frac{n(n-1)q\tilde{G}(s)}{2}$$

From the handshake theorem, the average node degree is

$$\bar{k} = (n-1)q\tilde{G}(s)$$

# Waxman in detail

Distribution of the lengths of edges $f(d\,|q,s)$ can be derived

$$
\begin{aligned}
f(d \mid q,s) &= \text{Prob}\left\{d_{ij} = d \mid (i,j) \in \mathcal{E}\right\} \\
&= \frac{\text{Prob}\left\{d_{ij} = d \;\&\; (i,j) \in \mathcal{E}\right\}}{\text{Prob}\left\{(i,j) \in \mathcal{E}\right\}} \\
&= \frac{\text{Prob}\left\{(i,j) \in \mathcal{E} \mid d_{(i,j)} = d; q,s\right\} \text{Prob}\left\{d_{ij} = d\right\}}{\text{Prob}\left\{(i,j) \in \mathcal{E} \mid q,s\right\}} \\
&= \frac{q\exp(-sd)g(d)}{\int_{t=0}^{\infty} q\exp(-st)g(t)\,dt} \\
&= \frac{g(d)\exp(-sd)}{\tilde{G}}
\end{aligned}
$$

# Waxman in detail

Average distance of edges

$$\begin{aligned}
\mathbb{E}[d \mid s] &= \frac{1}{\tilde{G}(s)} \int_0^\infty t g(t) \exp(-st)\, dt \\
&= -\frac{\tilde{G}'(s)}{\tilde{G}(s)}.
\end{aligned}$$

# Parameter estimation

The above result suggests a moment-based estimator

- We match the means to get $s$, *e.g.*, , measure

$$\hat{d} = \frac{1}{|E|} \sum_{e} d_{i,j},$$

  and find $\hat{s}$ such that

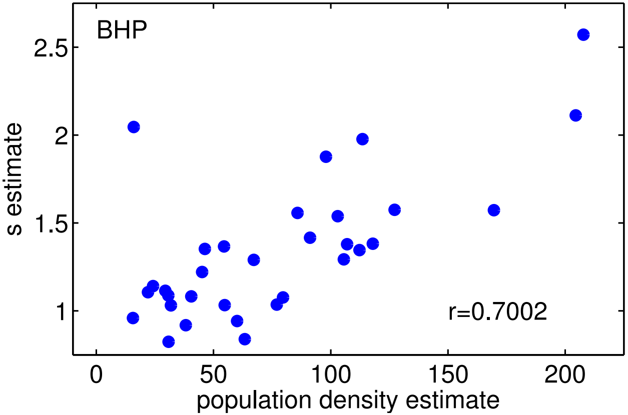
$$\frac{\tilde{G}'(s)}{\tilde{G}(s)} = -\hat{d}$$

- Estimate $q$ exactly how we did for Gilbert-Erdős-Rényi
  - convert it into estimate of Binomial parameter
- It turns out this is also the MLE (Maximum Likelihood Estimator)
- $|N|$, $|E|$ and $\bar{d}$ are sufficient statistics

# Example: voles [DAS+14]

- Large marked-capture-recapture experiment with voles
  - English-Scottish border over a 7-year period to study
  - Field voles *(Microtus agrestis)*
- Trap locations in a grid
  - traps emptied multiple times
  - contact between voles presumed if they were caught in same trap (on separate occasions)
  - generate multiple graphs at 4 different sites, and different time periods
- Fitted several models
  - their favoured model was similar to Waxman, but a little more complex
  - I have done a Waxman fit
  - most interesting is relationship between population density and $\hat{s}$

# Example: voles [DAS+14]




- As population density increases, voles travel shorter distances
- Has important consequences for disease transmission

Section 2

# Small-world graphs

# Six degrees of Kevin Bacon

Here's a game

- Pick an actor (any actor will do)
- Determine the shortest path to Kevin Bacon on the graph of movies in actors collaborate
- Famous result is that these paths rarely have more than 6 hops (6 degrees of separation).
  - *e.g.*, Nicole Kidman $\rightarrow$ Jeff Perry $\rightarrow$ Kevin Bacon
  - actually there are lots of cases for this particular connection

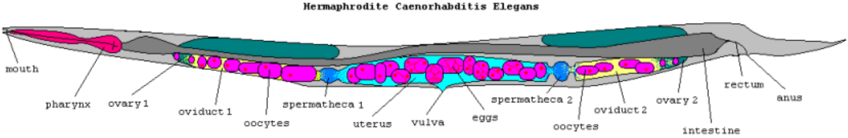
http://oracleofbacon.org/cgi-bin/movielinks

# Experiment 1: letter forwarding

Stanley Milgram [Mil67] experiment

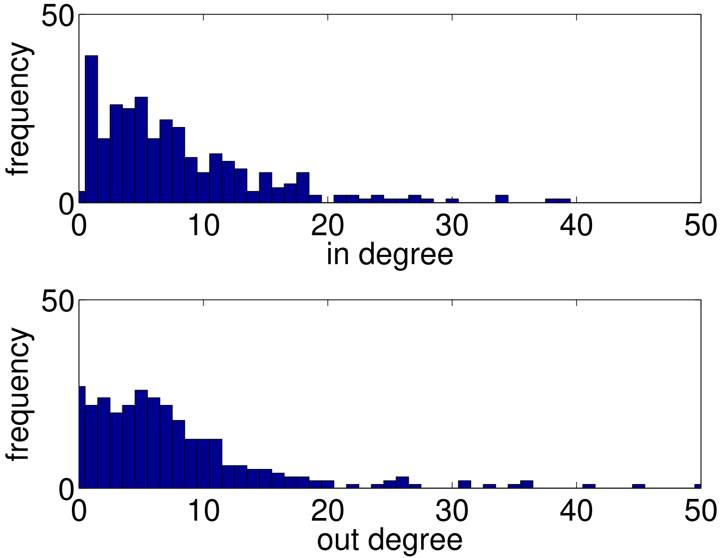
- gave people envelopes and the name of a target stranger
- mission: get envelope to destination
- method forward to someone they know (on 1st name basis)
- interesting result was how short the path-lengths were
  - the average was 6
  - 6-degrees of separation
- lots of problems with the study, but also lots of interest

# Experiment 2: *Caenorhabditis elegans*

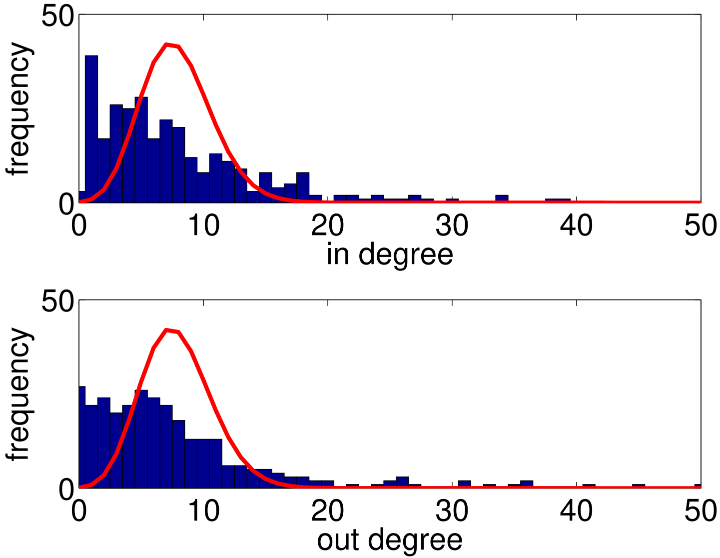



- *C. elegans* [SH77, SSWT83] is a small (∼1mm) soil nematode (worm)
- its very simple (only 959 somatic cells)
- its neural network was mapped in the 70's and 80's [AT76, WSNB86]
  - 302 neurons
  - database available
    `http://ims.dse.ibaraki.ac.jp/ccep/`

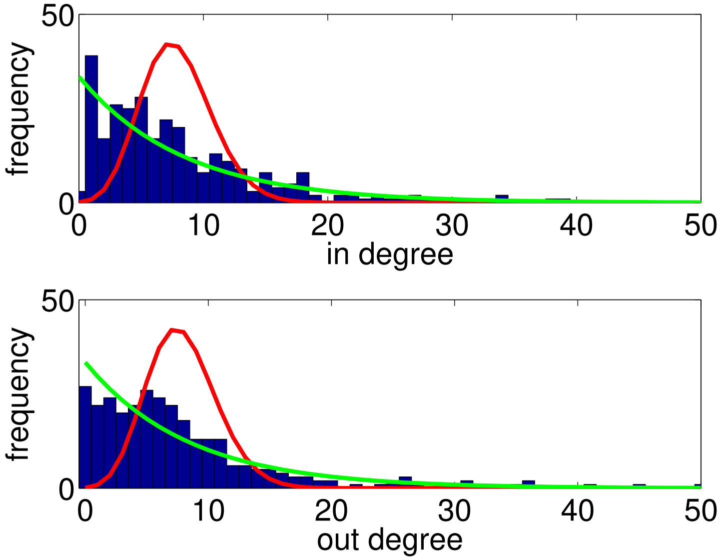
# Experiment 2: *C. elegans*



# Experiment 2: *C. elegans*



# Experiment 2: *C. elegans*



# Experiment 2: *C. elegans*

- mean path length 2.45
  - reported as 2.65 in [WS98]
  - much smaller than the size (nodes = 302) of the network
  - equivalent ER random network with same $n$ and $e$ mean distance is 2.34 (reported as 2.25 in [WS98])
- mean (local) clustering coefficient 0.31
  - reported as 0.28 in [WS98]
  - equivalent ER random network with same $n$ and $e$ mean distance is 0.055 (reported as 0.05 in [WS98])
- notably:
  - distances are similar
  - clustering is completely different

# Small world networks

- Watts and Strogatz [WS98] noted that many networks have two properties
  - short path distances
  - high clustering
- They proposed a model
  - start with a highly regular network with lots of clustering
  - rewire some links at random

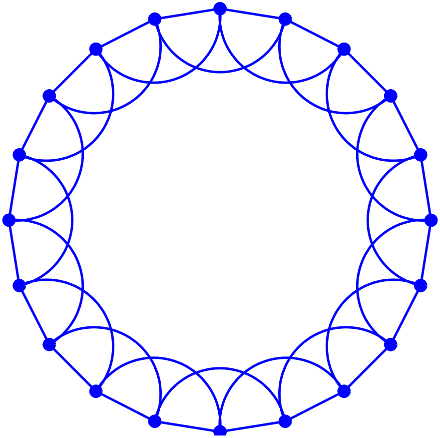
# Small world networks

Formally:

- start with a regular ring network
  - $n$ nodes on a circle, linked to $k$ nearest neighbours
  - $k$ even for it to be regular
- rewire each link with probability $p$
  - take an existing link, and send it to a random node

# Regular ring network

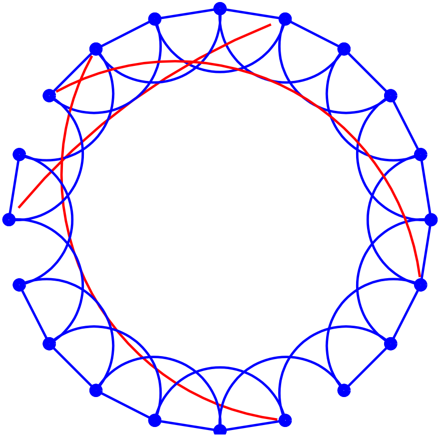
$n = 20$ and $k = 4$



$p = 0$

# Regular ring network

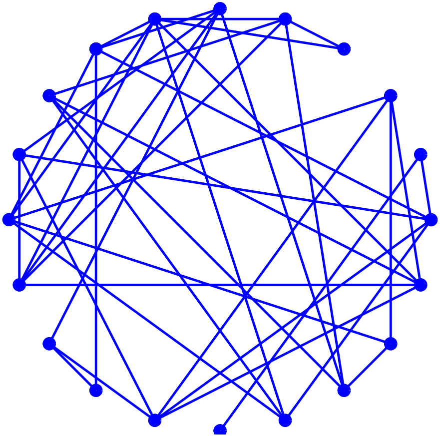
$n = 20$ and $k = 4$



$p = 0 \;\rightarrow$ increasing randomness

# Regular ring network

$n = 20$ and $k = 4$



$p = 0 \;\rightarrow$ increasing randomness $\rightarrow p = 1$

# Small world network clustering

**Theorem**

*The clustering coefficient for the $k$-regular ring (and $n > k$) is*

$$C = 1 - \frac{e(e-1)}{k(k-1)},$$

*where $k$ is the number of neighbours ($k$ even) and*

$$e = k/2 + 1.$$

*In the limit for large $k$ is $C \to 0.75$*

# Small world network clustering

## Proof.

We can derive the clustering coefficient for the regular ring as follows:

- WLOG start from node 1.
- Each node has $k$ neighbours, so the clustering coefficient will be

$$C = \frac{n_e}{k(k-1)/2}$$

  where $n_e$ is the number of links between the neighbours.
- It is easier to consider the number of missing links, so that

$$C = 1 - \frac{n_e^C}{k(k-1)/2}$$

  where $n_e^C$ is the number of missing links between neighbours.
- List the neighbours in consecutive order around the ring from "left" to "right".

# Small world network clustering

## continued.

- The first neighbour will connect to $k/2$ nodes (including node 1, which we ignore because it isn't a "neighbour"), *i.e.*, $k/2$ of the nodes. So it misses $k/2$ links.
- The second neighbour will connect to $k/2$ nodes to its right (including node 1), and one to the left, so it misses $k/2 - 1$ links.
- Continue until we get to the $k/2$ node (immediately to the left of node 1), and this will miss 1 link.
- Repeat the same argument from the other side of the original node, but remember to divide the total by 2 because links are undirected.
- The total number of missed links will be

$$\begin{aligned} n_e^C &= 2\left[(k/2) + (k/2 - 1) + \cdots + 1\right]/2 \\ &= (k/2+1)(k/2)/2 \\ &= e(e-1)/2. \end{aligned}$$

# Small world network clustering

Make clustering a function of $p$, the probability of rewiring

- ER random graph $C(1) \simeq 0$
- Regular ring (in limit) has $C(0) = 0.75$
- Small-world network $p \in (0,1)$ interpolates between these

# Small world networks path length

Distances in regular ring $(n, k)$:

- WLOG take an arbitrary start node, say 1
- The furtherest node away on the ring is distance $n/2$
- We can reach this node in steps of $k/2$
- So distance to this node is $\lceil n/k \rceil$ hops
- But we want an average over all nodes. They are equi-spaced around the ring, so where $n$ is divisible by $k$ a crude average can be obtained by dividing by two.

$$E[L] = \frac{n}{2k}$$

- However, we should take into account the fact that the last hop, won't be length $k/2$ for many of the intermediate nodes. So the path lengths will be slightly longer. The extra distance can be seen in the first hop, where dividing by $k/2$ would tell you less than one hop, but there is exactly one. It can be approximated by $(k/2 - 1)/(k/2)$.

# Small world networks path length

Make $L$ a function of $p$, the probability of rewiring

- ER random graph distances are fairly short (assuming connectivity)

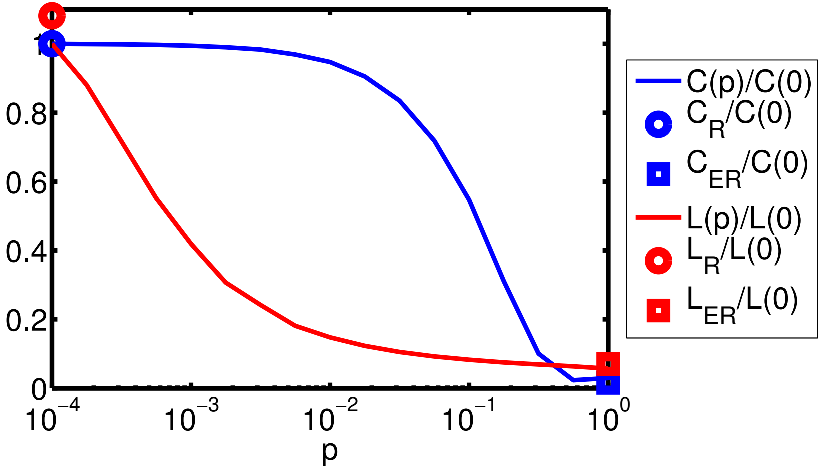
$$L(1) \simeq \log n / \log k$$

- Regular ring

$$L(0) \simeq n/2k + (k/2 - 1)/(k/2)$$

  where $k$ is the number of neighbours ($k$ even)
- Small-world network interpolates between these

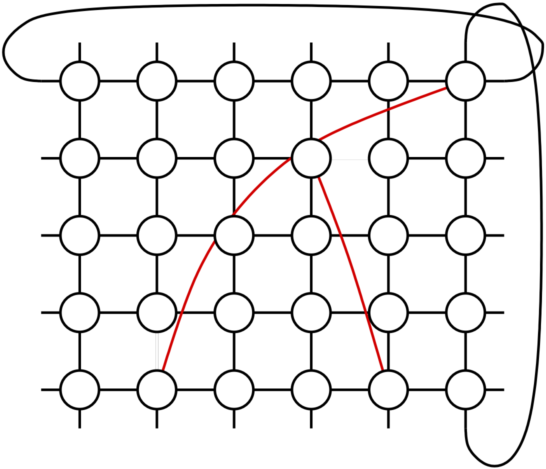
# Small world networks features



# Generalizations

Small world on a lattice

- a ring is 1D
- what about starting with a lattice on a torus



# Parameter estimation

- I haven't seen any formal literature on estimation
- Could hack up something that matches $\bar{L}$ and $\bar{C}$
- Do I think it's worth it?

# Further reading I


D. G. Albertson and J. N. Thomson, *The pharynx of caenorhabditis elegans*, Phil. Trans. R. Soc. London B **275** (1976), 299–325.

Chen Avin, *Distance graphs: from random geometric graphs to Bernoulli graphs and between*, Fifth International Workshop on Foundations of Mobile Computing (New York, NY, USA), DIALM-POMC '08, ACM, 2008, pp. 71–78.

Stephen Davis, Babak Abbasi, Shrupa Shah, Sandra Telfer, and Mike Begon, *Spatial analyses of wildlife contact networks*, Journal of The Royal Society Interface **12** (2014), no. 102, `http://rsif.royalsocietypublishing.org/content/12/102/20141004.short`.

P. Erdős and A. Rényi, *On the evolution of random graphs*, Publications of the Mathematical Institute of the Hungarian Academy of Sciences **5** (1960), 17–61.

Andras Farago, *Scalable analysis and design of ad hoc networks via random graph theory*, Dial-M'02, 2002.

# Further reading II


- Cigdem Gunduz-Demir, *Mathematical modeling of the malignancy of cancer using graph evolution*, Mathematical Biosciences **209** (2007), no. 2, 514–527.
- E.N. Gilbert, *Random graphs*, Annals of Mathematical Statistics **30** (1959), 1441–1144.
- E. N. Gilbert, *Random plane networks*, Journal of the Society for Industrial and Applied Mathematics **9** (1961), no. 4, 533–543.
- Peter Grindrod, *Range-dependent random graphs and their application to modeling large small-world proteome datasets*, Phys. Rev. E **66** (2002), no. 066702.
- R. Hekmat and P. Van Mieghem, *Degree distribution and hopcount in wireless ad-hoc networks*, Networks, 2003. ICON2003, 2003, pp. 603–609.

# Further reading III


- Stanley Milgram, *The small world problem*, Psychology Today **1** (1967), no. 1, 60–67.
- J. E. Sulston and H. R. Horvitz, *Post-embryonic cell lineages of the nematode Caenorhabditis elegans*, Developmental Biology **56** (1977), 110–156.
- J. E. Sulston, E. Schierenberg, J. G. White, and J. N. Thomson, *The embryonic cell lineage of the nematode Caenorhabditis elegans*, Developmental Biology **100** (1983), 64–119.
- B.M. Waxman, *Routing of multipoint connections*, IEEE J. Select. Areas Commun. **6** (1988), no. 9, 1617–1622.
- Duncan J. Watts and Steven H. Strogatz, *Collective dynamics of 'small-world' networks*, Letters to Nature **393** (1998), no. 4, 440–442.

# Further reading IV


J. G. White, E. Southgate, Thomson J. N., and S. Brenner, *The structure of the nervous system of the nematode Caenorhabditis elegans*, Phil. Trans. R. Soc. London B **314** (1986), 1–340.

Ellen W. Zegura, Ken Calvert, and S. Bhattacharjee, *How to model an internetwork*, IEEE Infocom (San Francisco, CA), 1996, pp. 594–602.

Ellen W. Zegura, Kenneth L. Calvert, and Michael J. Donahoo, *A quantitative comparison of graph-based models for Internet topology*, IEEE/ACM Transactions on Networking **5** (1997), no. 6, 770–783.