Optimisation and Operations Research

Lecture 6: Simplex Phase II

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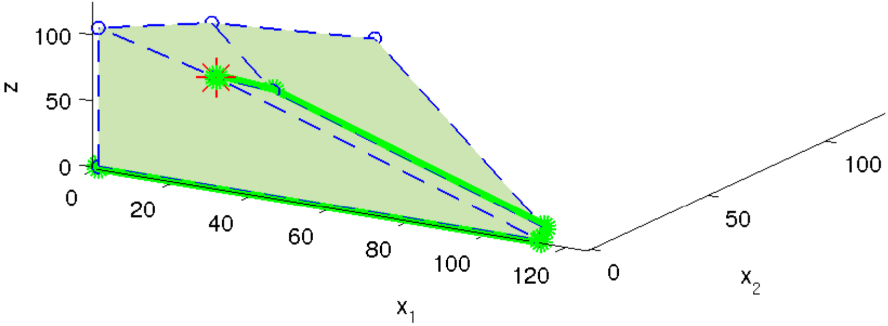
Section 1

Simplex Phase II

Intuition

- Assume we start in feasible, canonical form
    - Corresponds to a feasible vertex in original inequality form
    - Often starts this way
    - Simplex Phase I will fix this if it isn't already
- Test to see if it is optimal (if so then stop)
- Move to an *adjacent* vertex such that
    - the solution remains feasible
    - the solution gets better (or at least no worse)
- And repeat until its optimal, or we can't go any further



$$
\begin{array}{rrcrcrcrcl}
\max & z & = & 13x_1 & + & 12x_2 & + & 17x_3 & & \\
\text{subject to} & & & 2x_1 & + & x_2 & + & 2x_3 & \leq & 225 \\
 & & & x_1 & + & x_2 & + & x_3 & \leq & 117 \\
 & & & 3x_1 & + & 3x_2 & + & 4x_3 & \leq & 420 \\
 & & & & & & & x_i & \geq & 0, \qquad \text{for } i = 1, 2, 3
\end{array}
$$

Operating on a tableau

Imagine the Simplex Tableau

$$M = \begin{array}{|c|c|c|} \hline A & \mathbf{0} & \mathbf{b} \\ \hline -\mathbf{c}^T & 1 & z_0 \\ \hline \end{array}$$

If we perform the same row operations we were allowed in Gauss-Jordan,

$$M \overset{rowops}{\rightarrow} M'$$

then M' will be an *equivalent tableau*

- it expresses exactly the same LP
- for exactly the same reasons that G-J works

Pivots

- We can perform the same row operations we were allowed in Gauss-Jordan.
- We want to change vertices:
  - change which constraints are active (in inequality form)
  - change which variables are basic (in equality form)
- We do this with *pivot* operations on (i, j) where
  - i means the ith row of A
  - j means the jth column of A

  We don't pivot in the z column because it represents the objective.

One pivot (plus associated bits) is one *iteration* of Simplex

$$M^{(i)} \rightarrow M^{(i+1)}$$

Entering and Leaving Variables

When we perform a pivot, one variable *leaves* the set of basic variables, and another *enters* the basic set.

Definition (Entering Variable)

The entering variable is the variable that enters the basic set after a pivot operation. It corresponds to the column j of the pivot.

Definition (Leaving Variable)

The leaving variable is the variable that leaves the basic set after the pivot operation. The row i of the pivot determines the leaving variable, by selecting constraint i, and the current basic variable k associated with it.

Rules for Choosing Entering and Leaving Variables

There are different rules for choosing the entering and leaving variables, but these rules have two components:

- *mandatory*: all choices must satisfy these in order for Simplex to work
- *discretionary*: choices can pick and choose between these. They matter, because may affect the performance of the overall algorithm, and sometimes will have a big impact, but there are alternatives.

Choosing the Entering Variable (the column)

- *Mandatory:* the choice of column j must satisfy

  $$-c_j < 0,$$

  i.e., the current jth value in the z-row must be negative.
- *Discretionary:* when there is a choice between multiple possible negative values, we can implement several different approaches.
  - *e.g.*, choose the most negative

I will tell you the mandatory rules, but I *want you to work out what discretionary rules makes the most sense.*

Choosing the Entering Variable: Example

Example

		pivot column					
x_1	x_2	x_3	x_4	x_5	x_6	z	b
1	3	-2	1	0	0	0	3
-1	2	1	0	1	0	0	1
0	3	1	0	0	1	0	2
2	1	-2	0	0	0	1	-2

We choose $j = 3$ because this is the only A-column with $-c_j < 0$

Choosing the Leaving Variable (pivot row)

- *Mandatory:* given column j, the choice of row i must
  - $a_{ij} > 0$
  - of the non-negative possibilities, choose smallest possible b_i/a_{ij}
- *Discretionary:* when there is more than one equal value of b_i/a_{ij} we get to choose one.

I want you to work out what discretionary rule makes the most sense.

Choosing the Leaving Variable: Example

Example

x_1	x_2	x_3	x_4	x_5	x_6	z	b		
1	3	-2	1	0	0	0	3	$a_{ij} < 0$	
-1	2	1	0	1	0	0	1	$b_i/a_{ij} = 1$	pivot row
0	3	1	0	0	1	0	2	$b_i/a_{ij} = 2$	
2	1	-2	0	0	0	1	-2		

We already chose $j = 3$, so now examine possible rows,

- we can eliminate row 1, because $a_{13} \leq 0$
- we have

$$\begin{aligned} b_2/a_{23} &= 1 \\ b_3/a_{33} &= 2 \end{aligned}$$

  we choose the smallest, so $i = 2$

- $\ell_B = \{4, 5, 6\}$, and the ith gives us the leaving variable $x_{\ell_2} = x_5$

Stopping rules

- If you can't satisfy a mandatory rule, then Simplex stops
- Output state:
  - if you can't find a new entering variable, *i.e.*, no column has $-c_j < 0$ then you have the optimal value
  - if you can't find a new leaving variable, then the problem is *unbounded*
- Why do the choice rules work?
  - we have two imperatives
    1. stay feasible
    2. improve the objective function (or at least make it no worse)
  - the mandatory rules are aimed at these imperatives

Why Have Mandatory Choice Rules: column

Why do we require $-c_j < 0$ for the pivot column j?

- Simplex keeps track of z-value in z_0 (bottom right corner)
- When we pivot at (i, j)

$$z_0' = z_0 + \frac{c_j}{a_{ij}} b_i$$

  - Feasible canonical form has $b_i \geq 0$
  - We choose a pivot so $a_{ij} > 0$
  - If $-c_j < 0$, then the above means that

$$z_0' \geq z_0$$

- This rule ensures that the *objective is non-decreasing*
  - it only stays the same if $b_i = 0$
- We stop when we can't find such a column, because the objective can't be increased any more, *i.e.*, we have found the optimal solution

Why Have Mandatory Choice Rules: row

Why do we require $a_{ij} > 0$ and smallest possible b_i/a_{ij} for row i?

- (1) so objective remains non-decreasing
- (2) when we pivot at (i, j) the new $\mathbf{b}$ column is

  $$b'_k = b_k - \frac{a_{kj}}{a_{ij}} b_i, \quad \text{for } i \neq j$$

  for *feasible* canonical form $b_k \geq 0$, and we choose $a_{ij} > 0$ so

  - if $a_{kj} \leq 0$ then $b'_k \geq b_k \geq 0$
  - if $a_{kj} > 0$ for all rows $k \neq i$

    $$b'_k = a_{kj} \left(\frac{b_k}{a_{kj}} - \frac{b_i}{a_{ij}} \right) \geq 0$$

    so $b'_k \geq 0$ we choose i such that for all rows $k \neq i$

    $$\frac{b_k}{a_{kj}} \geq \frac{b_i}{a_{ij}}$$

- So this rule maintains *feasibility*

Why Have Mandatory Choice Rules: row

What if we can't find a row?

- That means all of $a_{ij} \leq 0$.
- Consider the following solution
  - set entering variable $x_j = t > 0$
  - set other free variables to 0
  - solve for the basic variables ℓ_B

$$x_{\ell_i} = b_i - a_{ij} t$$

    - ★ normally, we would have had $t = 0$, and $x_{\ell_i} = b_i$
    - ★ note $x_{\ell_i} \geq 0$ as $b_i \geq 0$, $t > 0$ and $a_{ij} < 0$
    - ★ so this solution is feasible for any value $t > 0$
- Hence we have an *arbitrarily large* feasible solution
- The problem is *unbounded*

Discretionary Choice Rules

I want you to find the *discretionary* part of your choice rules.

- research them – you may want to Google “Bland’s rule”
- you need to write a `Simplex` program using functions so that you can change choice rules easily and experiment (some class exercises may require you to be able to try different choice rules).

You will be examined on choice rules, so this part of your work is not optional!!!

Output

Finally, if we found a feasible, optimal value, just read off the solution

1. All of the non-basic variables are zero.
2. For the basic variables in the list ℓ_B, just read of the values

$$x^*_{\ell_B(i)} \leftarrow b_i$$

3. z^*, the optimal value of the objective function, is given by the bottom value of the **b** column (where you originally placed z_0)

Simplex Phase II pseudo-code

Assumes a problem in standard equality form: $\max_{\mathbf{x}} \left\{ z = \mathbf{c}^T\mathbf{x} + z_0 \;\middle|\; A\mathbf{x} = \mathbf{b}, \mathbf{x} \geq 0 \right\}$ such that when we construct the Tableau M it will be in feasible canonical form

```
Input: $A$, $\mathbf{b}$, $\mathbf{c}$, $z_0$, $\ell_B$     // implicitly n variables, m constraints
Output: $\mathbf{x}^*$, $z^*$
Construct Simplex Tableau $M$     // an $(m+1) \times (n+2)$ matrix
while at least one $-c_i < 0$ do
    Select Entering Variable $x_j$ (column $j$)
    Select Leaving Variable $x_k$ (row $i$)
    if there is no possible leaving variable then
        break     // problem is unbounded
    else
        $\ell_B(i) \leftarrow j$
        $M \leftarrow \text{Pivot}(M, i, j)$
    end
end
Set $x^* = \underline{0}$
for $i = 1 \ldots m$ do
    $x^*_{\ell_B(i)} \leftarrow b_i = M(i, \text{b-col})$
end
$z^* \leftarrow M(\text{z-row}, \text{b-col})$
```

Algorithm 1: Simplex Phase II

Simplex Phase II Example: iteration 1

Starts at $\mathbf{x}_0 = (0, 0, 0, 225, 117, 420)$, $z_0 = 0$

x_1	x_2	x_3	s_4	s_5	s_6	z	b
2	1	2	1	0	0	0	225
1	1	1	0	1	0	0	117
3	3	4	0	0	1	0	420
-13	-12	-17	0	0	0	1	0

$$\Downarrow$$

1	$1/2$	1	$1/2$	0	0	0	$225/2$
0	$1/2$	0	$-1/2$	1	0	0	$9/2$
0	$3/2$	1	$-3/2$	0	1	0	$165/2$
0	$-11/2$	-4	$13/2$	0	0	1	$2925/2$

After the pivot $\mathbf{x}_1 = (225/2, 0, 0, 0, 9/2, 165/2)$, $z_1 = 1462.5$

Simplex Phase II Example: iteration 2

Starts at $\mathbf{x}_1 = (225/2, 0, 0, 0, 9/2, 165/2)$, $z_1 = 1462.5$

x_1	x_2	x_3	s_4	s_5	s_6	z	b
1	$1/2$	1	$1/2$	0	0	0	$225/2$
0	$1/2$	0	$-1/2$	1	0	0	$9/2$
0	$3/2$	1	$-3/2$	0	1	0	$165/2$
0	$-11/2$	-4	$13/2$	0	0	1	$2925/2$

$$\Downarrow$$

1	0	1	1	-1	0	0	108
0	1	0	-1	2	0	0	9
0	0	1	0	-3	1	0	69
0	0	-4	1	11	0	1	1512

After the pivot $\mathbf{x}_2 = (108, 9, 0, 0, 0, 69)$, $z_2 = 1512$

Simplex Phase II Example: iteration 3

Starts at $\mathbf{x}_2 = (108, 9, 0, 0, 0, 0, 69)$, $z_2 = 1512$

x_1	x_2	x_3	s_4	s_5	s_6	z	b
1	0	1	1	-1	0	0	108
0	1	0	-1	2	0	0	9
0	0	1	0	-3	1	0	69
0	0	-4	1	11	0	1	1512

$$\Downarrow$$

1	0	0	1	2	-1	0	39
0	1	0	-1	2	0	0	9
0	0	1	0	-3	1	0	69
0	0	0	1	-1	4	1	1788

After the pivot $\mathbf{x}_3 = (39, 9, 69, 0, 0, 0)$, $z_3 = 1788$

Simplex Phase II Example: iteration 4

Starts at $\mathbf{x}_3 = (39, 9, 69, 0, 0, 0)$, $z_3 = 1788$

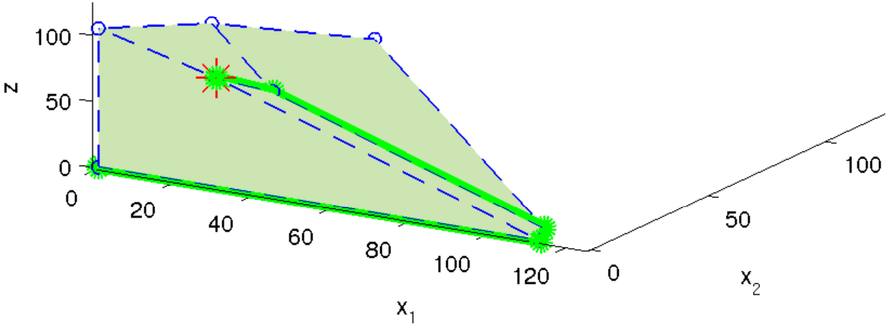
x_1	x_2	x_3	s_4	s_5	s_6	z	b
1	0	0	1	2	-1	0	39
0	1	0	-1	2	0	0	9
0	0	1	0	-3	1	0	69
0	0	0	1	-1	4	1	1788

$$\Downarrow$$

1	-1	0	2	0	-1	0	30
0	$1/2$	0	$-1/2$	1	0	0	$9/2$
0	$3/2$	1	$-3/2$	0	1	0	$165/2$
0	$1/2$	0	$1/2$	0	4	1	$3585/2$

After the pivot $\mathbf{x}_4 = (30, 0, 165/2, 0, 9/2, 0)$, $z_4 = 1792.5$

- there are no more possible pivot columns, so we stop, and this is the optimal solution


z
100
50
0
0
20
40
60
80
100
120
x_1
0
50
100
x_2

Takeaways

- Simplex Phase II
  - starts in feasible canonical form
  - operates by choosing pivots (rules)
  - each pivot
    - keeps it feasible
    - makes objective no worse
  - ends
    - unbounded
    - optimal solution

Further reading I