Optimisation and Operations Research

Lecture 9: Duality and Complementary Slackness

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Section 1

Duality

Problem Recap

We will start with a problem in *standard equality form*.

$$\begin{array}{rrcl} \max & z & = & \mathbf{c}^T\mathbf{x} + z_0 \\ \text{such that} & A\mathbf{x} & = & \mathbf{b} \\ \text{and} & \mathbf{x} & \geq & 0 \end{array}$$

We call this the *primal* LP

Dual

Primal *(P)*

$$\begin{array}{rrcl} \max & z & = & \mathbf{c}^T\mathbf{x} + z_0 \\ \text{such that} & A\mathbf{x} & = & \mathbf{b} \\ \text{and} & \mathbf{x} & \geq & 0 \end{array}$$

Consider a new LP called the *dual* problem *(D)*

$$\begin{array}{rrcl} \min & w & = & \mathbf{b}^T\mathbf{y} + z_0 \\ \text{such that} & A^T\mathbf{y} & \geq & \mathbf{c} \\ \text{and} & \mathbf{y} & \textit{free} & \end{array}$$

Origin of the dual

Suppose there is an optimal solution $(\mathbf{x}^*, z^*)$ to the primal LP

$$\begin{array}{rrcl} \max & z & = & \mathbf{c}^T\mathbf{x} + z_0 \\ \text{such that} & A\mathbf{x} & = & \mathbf{b} \\ \text{and} & \mathbf{x} & \geq & 0 \end{array}$$

with $z^* = \mathbf{c}^T\mathbf{x}^* + z_0$.

We might have obtained this via Simplex

Initial Tableau $\overset{\textit{Simplex}}{\Longrightarrow}$ **Final Tableau**

Initial Tableau:

A	$\mathbf{b}$
$-\mathbf{c}^T$	z_0

Final Tableau:

$\hat{A}$	$\hat{\mathbf{b}}$
$-\hat{\mathbf{c}}^T$	$\hat{z}_0$

where $z^* = \hat{z}_0$

Origin of the dual

The final tableau comes from pivots, so there are some numbers $y_1^*, \ldots, y_m^*$ such that

$$
\begin{aligned}
-\hat{c}_j &= \sum_{i=1}^{m} y_i^* a_{ij} - c_j, \quad j = 1, \ldots, n \\
\hat{z}_0 &= \sum_{i=1}^{m} y_i^* b_i + z_0 \ .
\end{aligned}
$$

At the end of Simplex $-\hat{c}_j \geq 0$ so

$$
\sum_{i=1}^{m} y_i^* a_{ij} \geq c_j \quad \text{for } j = 1, \ldots, n.
$$

So let's consider any variables $y_1, \ldots, y_m$ which satisfy

$$
\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad \text{for} \quad j = 1, \ldots, n.
$$

We could look for an optimisation to find the $\mathbf{y}^*$

Origin of the dual

Let's build a new objective function with

$$w = \sum_{i=1}^{m} y_i b_i + z_0.$$

From the Primal, $b_i = \sum_{j=1}^{n} a_{ij} x_j$ so

$$\begin{aligned} w &= \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) + z_0 \\ &= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j + z_0 \\ &\geq \sum_{j=1}^{n} c_j x_j + z_0 \end{aligned}$$

Hence $w \geq z$, but also the above is true for any feasible $\mathbf{x}$, so $w \geq \hat{z}_0$, the optimal value of the primal. But we defined w so that $w(\mathbf{y}^*) = \hat{z}_0$, so $\hat{z}_0$ is the minimum of w.

The Dual of an LP (Summary)

In the dual, variables become constraints, and visa versa

(P) $\max z = \sum_{j=1}^{n} c_j x_j + z_0$	(D) $\min w = \sum_{i=1}^{m} y_i b_i + z_0$
$\sum_{j=1}^{n} a_{ij} x_j = b_i, \quad i = 1, \ldots, m$	y_i free, $\quad i = 1, \ldots, m$
$x_j \geq 0, \quad j = 1, \ldots, n$	$\sum_{i=1}^{m} y_i a_{ij} \geq c_j, \quad j = 1, \ldots, n$
or	or
$\max \quad z = \mathbf{c}^T \mathbf{x} + z_0,$	$\min \quad w = \mathbf{y}^T \mathbf{b} + z_0,$
$A\mathbf{x} = \mathbf{b},$	$\mathbf{y}^T A \geq \mathbf{c}^T,$
$\mathbf{x} \geq 0.$	$\mathbf{y}$ free.

Note how the lines are paired up

Dual Example

Example (Dual)

$$
\begin{array}{llrcrcrcr}
(P) & \max\ z = & 3x_1 & - & x_2 & + & x_3 & & \\
\text{s.t.} & & x_1 & + & 2x_2 & & & = & 5 \\
& & x_1 & - & x_2 & + & x_3 & = & 7 \\
& & & & x_2 & + & 6x_3 & = & 11
\end{array}
$$

$$x_1 \geq 0,\ x_2 \geq 0,\ x_3 \geq 0.$$

$$
\begin{array}{llrcrcrcr}
(D) & \min\ w = & 5y_1 & + & 7y_2 & + & 11y_3 & & \\
\text{s.t.} & & y_1 & + & y_2 & & & \geq & 3 \\
& & 2y_1 & - & y_2 & + & y_3 & \geq & -1 \\
& & & & y_2 & + & 6y_3 & \geq & 1
\end{array}
$$

$y_1,\ y_2$ and y_3 are all free variables.

Why do we call it the dual?

Theorem

The Dual of the Dual is the Primal.

Proof: The dual is

$$\begin{array}{rrcl} \min & w & = & \mathbf{b}^T\mathbf{y} + z_0 \\ \text{such that} & A^T\mathbf{y} & \geq & \mathbf{c} \\ \text{and} & \mathbf{y} & \textit{free} & \end{array}$$

We convert to standard equality form by

- multiplying the objective by -1 (to make it a max)
- multiplying the constraints by -1
- adding slack variables
- replacing free variables y_i with non-negative variables $y_i^+ \geq 0$ and $y_i^- \geq 0$ such that $y_i = y_i^+ - y_i^-$,

Why do we call it the dual?

Proof: (continued) The dual, written in standard equality form is (D')

$$\begin{array}{rrcl} \text{max} & -w & = & \mathbf{b}^T\mathbf{y}^- - \mathbf{b}^T\mathbf{y}^+ - z_0 \\ \text{such that} & A^T\mathbf{y}^- - A^T\mathbf{y}^+ + \mathbf{s} & = & -\mathbf{c} \\ \text{and} & \mathbf{y}^+, \mathbf{y}^-, \mathbf{s} & \geq & 0 \end{array}$$

Now we can take the dual of (D') which we denote (DD) by creating a variable x_i for each constraint, and a constraint for each variable y_j and s_j.

$$\begin{array}{rrcll} \text{min} & u & = & -\mathbf{c}^T\mathbf{x} - z_0 & \\ \text{such that} & A\mathbf{x} & \geq & \mathbf{b}^T & \text{from } \mathbf{y}^- \\ \text{and} & -A\mathbf{x} & \geq & -\mathbf{b}^T & \text{from } \mathbf{y}^+ \\ \text{and} & \mathbf{x} & \geq & 0 & \text{from } \mathbf{s} \end{array}$$

Why do we call it the dual?

Proof: (continued) We have (DD)

$$\begin{array}{rrcll} \min & u & = & -\mathbf{c}^T\mathbf{x} - z_0 & \\ \text{such that} & A\mathbf{x} & \geq & \mathbf{b}^T & \text{from } \mathbf{y}^- \\ \text{and} & -A\mathbf{x} & \geq & -\mathbf{b}^T & \text{from } \mathbf{y}^+ \\ \text{and} & \mathbf{x} & \geq & 0 & \text{from } \mathbf{s} \end{array}$$

Note that

- we can multiply the objective by -1 (to have a max)
- The constraints $A\mathbf{x} \geq \mathbf{b}^T$ and $A\mathbf{x} \leq \mathbf{b}^T$ together imply that $A\mathbf{x} = \mathbf{b}$.

So (DD) $\equiv$ (P).

Q.E.D.

Weak Duality

$$
\begin{array}{rclcrcl}
 & \text{Primal} & & & & \text{Dual} & \\
\max z & = & \mathbf{c}^T\mathbf{x} + z_0 & & \min w & = & \mathbf{b}^T\mathbf{y} + z_0 \\
A\mathbf{x} & = & \mathbf{b} & & A^T\mathbf{y} & \geq & \mathbf{c} \\
\mathbf{x} & \geq & 0 & & \mathbf{y} & \textit{free} &
\end{array}
$$

Theorem

Given primal and dual problems as above have feasible solutions $\mathbf{x}$ *and* $\mathbf{y}$*, then* $z \leq w$.

Weak Duality Proof

Proof: We essentially showed this earlier:

$$
\begin{aligned}
w &= \sum_{i=1}^{m} y_i b_i + z_0 \\
&= \sum_{i=1}^{m} y_i \left(\sum_{j=1}^{n} a_{ij} x_j \right) + z_0 && \text{because (P) requires } b_i = \sum_{j=1}^{n} a_{ij} x_j \\
&= \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} \right) x_j + z_0 && \text{swapping order of summation} \\
&\geq \sum_{j=1}^{n} c_j x_j + z_0 && \text{from constraints of (D)} \\
&= z
\end{aligned}
$$

Hence $w \geq z$

Weak Duality Consequences

Corollary

If we have feasible solutions $\mathbf{x}$ *and* $\mathbf{y}$ *to the primal and dual respectively and* $w = z$*, then these are optimal solutions to their respective problems.*

Proof: w is an upper bound on z (and visa versa), so if $z = w$ for a feasible solution, it has achieved its upper bound, and hence we have an optimal solution.

Q.E.D.

Boundedness v Feasibility

Corollary

If the primal (dual) problem is unbounded then the dual (primal) problem is infeasible.

Proof: If the primal were feasible and unbounded, then that means there can be no upper bound on z, so we cannot have a feasible solution the dual.
Similarly if the dual is unbounded.

Q.E.D.

Strong Duality

Theorem

If the primal (dual) problem has a finite optimal solution, then so does the dual (primal) problem, and these two values are equal.

Proof: see later as a result of complementary slackness.

The Dual of an LP – 12

Summary of Results for Primal/Dual pair (P) and (D)

1. For a feasible solution $x_1, \ldots, x_n$ of (P), with value z, and a feasible solution $y_1, \ldots, y_m$ of (D), with value w, we have $w \geq z$ (**Weak Duality**)

2. If (P) has an optimal solution then (D) has an optimal solution and max $z =$ min w (**Strong Duality**)

3. Because the dual of the dual is the primal, if (D) has an optimal solution then (P) has an optimal solution and max $z =$ min w (**Strong Duality**)

4. If (P) has no optimal solution ($z \to \infty$) then (D) cannot have a feasible solution (as $w \geq \max z$), and if (D) has no optimal solution ($w \to -\infty$) then (P) cannot have a feasible solution (as $z \leq \min w$).

The Dual of an LP – 13

Summary of Results for Primal/Dual pair (P) and (D)

		Primal (P)		
		stop 1 optimal solution	**stop 2** feasible sol^n, no opt. sol^n	**stop 3** no feasible solution
Dual (D)	**stop 1** optimal solution	possible ‡	impossible	impossible
	stop 2 feas. sol^n, no opt. sol^n	impossible	impossible	possible
	stop 3 no feasible solution	impossible	possible	possible

‡ $(\max z = \min w)$

Section 2

Complementary Slackness

Complementary slackness

Theorem (Complementary slackness)

Given primal problem P and dual problem D with feasible solutions $\mathbf{x}$ *and* $\mathbf{y}$*, respectively, then* $\mathbf{x}$ *is an optimal solution of* (P) *and* $\mathbf{y}$ *an optimal solution of* (D) *if and only if*

$$x_j \left(\sum_{i=1}^{m} y_i a_{ij} - c_j \right) = 0, \quad \text{for } \; j = 1, \ldots, n.$$

Implicit in this theorem is the Strong Duality Theorem, which we prove now, and the second part is called the *complementary slackness* relations.

Definition (Complementary Slackness Relations)

The relations $x_j \left(\sum_{i=1}^{m} y_i a_{ij} - c_j \right) = 0$, for each $j = 1, \ldots, n$ are called the *Complementary Slackness Relations* (CSR).

Complementary slackness proof

Proof: We have already seen the argument:

$$z = \sum_{j=1}^{n} c_j x_j + z_0 \leq \sum_{j=1}^{n} \sum_{i=1}^{m} y_i a_{ij} x_j + z_0 \qquad \left(\text{as } \sum_{i=1}^{m} y_i a_{ij} \geq c_j, \;\; x_j \geq 0 \right)$$

$$= \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_j y_i + z_0$$

$$= \sum_{i=1}^{m} y_i b_i + z_0 = w \qquad \left(\text{as } \sum_{j=1}^{n} a_{ij} x_j = b_i \right)$$

So $z \leq w$ for all feasible solutions

Complementary slackness proof

Proof : (cont)

From Strong Duality, if we have an optimal solution, then $z^* = w^*$. Reversing the above logic, we get

$$\sum_{j=1}^{n} c_j x_j + z_0 = \sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} x_j \right) + z_0$$

which, when we group the x_i terms together gives

$$\sum_{j=1}^{n} \left(\sum_{i=1}^{m} y_i a_{ij} - c_j \right) x_j = 0$$

Complementary slackness proof

Proof : (cont)

If we know that $n_j \geq 0$, and we have

$$\sum_j n_j = 0$$

then we must have $n_j = 0$.
This applies here because we know from the LP dual and primals that

$$\sum_{i=1}^{m} y_i a_{ij} - c_j \geq 0 \text{ and } x_j \geq 0.$$

Hence

$$\left(\sum_{i=1}^{m} y_i a_{ij} - c_j \right) x_j = 0, \text{ for each } j = 1, \ldots, n.$$

$\square$

Complementary Slackness: Example

Primal Tableax (P)

x_1	x_2	x_3	x_4	x_5	b
2	-1	-1	1	0	1
2	2	1	-2	0	3
1	1	1	1	1	2
-1	1	1	1	0	0

Simplex Phase I and II

1	0	$-\frac{1}{6}$	0	0	$\frac{5}{6}$
0	1	$\frac{2}{3}$	-1	0	$\frac{2}{3}$
0	0	$\frac{1}{2}$	2	1	$\frac{1}{2}$
0	0	$\frac{1}{6}$	2	0	$\frac{1}{6}$

Resulting solution $\mathbf{x}^* = (5/6, 2/3, 0, 0, 1/2)$, with $z^* = 1/6$.

Complementary Slackness: Example

Primal Tableax (P)

x_1	x_2	x_3	x_4	x_5	b
2	−1	−1	1	0	1
2	2	1	−2	0	3
1	1	1	1	1	2
−1	1	1	1	0	0

$\Rightarrow$

Dual Tableax (D)
(note that these represent $\geq$)

y_1	y_2	y_3	c
2	2	1	1
−1	2	1	−1
−1	1	1	−1
1	−2	1	−1
0	0	1	0
−1	−3	−2	0

Complementary slackness

Write down the dual (D) of (P).

$$
\begin{array}{llrcrcrcrl}
(D) & \min\ w & = & y_1 & + & 3y_2 & + & 2y_3 & & \\
 & & & 2y_1 & + & 2y_2 & + & y_3 & \geq 1 & \text{(iv)} \\
 & & - & y_1 & + & 2y_2 & + & y_3 & \geq -1 & \text{(v)} \\
 & & - & y_1 & + & y_2 & + & y_3 & \geq -1 & \text{(vi)} \\
 & & & y_1 & - & 2y_2 & + & y_3 & \geq -1 & \text{(vii)} \\
 & & & & & & & y_3 & \geq 0 & \text{(viii)} \\
 & & & \multicolumn{6}{l}{y_1,\ y_2 \text{ and } y_3 \text{ are all free variables}^1.} & \text{(i)-(iii)}
\end{array}
$$

Writing down the CSR, we see that

- From (iv), we get $(2y_1^* + 2y_2^* + y_3^* - 1)\, x_1^* = 0$ and since $x_1^* > 0$, we have $2y_1^* + 2y_2^* + y_3^* = 1$.
- From (v), since $x_2^* > 0$, we have $-y_1^* + 2y_2^* + y_3^* = -1$.
- Similarly, from (viii), since $x_5^* > 0$, we have $y_3^* = 0$.

[1] Note the last constraint tightens y_3, but this is OK.

Complementary slackness

Thus from the CSR for (iv),(v) and (viii) we see that

$$\begin{aligned} 2y_1^* + 2y_2^* + y_3^* &= 1 \\ -y_1^* + 2y_2^* + y_3^* &= -1 \\ y_3^* &= 0 \end{aligned}$$

Solving these equalities for y_1^*, y_2^*, y_3^*, we get $y_1^* = \dfrac{2}{3},\ y_2^* = -\dfrac{1}{6},\ y_3^* = 0$, which need to be checked for feasibility in the other constraints: That is,

$$\text{(vi)} \qquad -\tfrac{2}{3} - \tfrac{1}{6} + 0 \quad > \quad -1$$

$$\text{(vii)} \qquad \tfrac{2}{3} + \tfrac{1}{3} + 0 \quad > \quad -1.$$

Also note that $\boxed{z^* = \dfrac{5}{6} - \dfrac{2}{3} = \dfrac{1}{6}} \quad = \quad \boxed{w^* = \dfrac{2}{3} - 3 \times \dfrac{1}{6} = \dfrac{2}{3} - \dfrac{1}{2} = \dfrac{1}{6}}$.

Takeaways

- Optimisation problems have a Dual
  - this is a very general concept, and holds beyond LPs
- Complementary slackness relates dual solutions to primal
  - these give us an edge in knowing when we have found optimal solutions
  - we'll use these later!

Further reading I