Optimisation and Operations Research

Lecture 13: Complexity and the P vs NP problem

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Section 1

Turing machines

The Problem

- Earlier analysis essentially ignored the underlying computer
- Computation time depends a great deal on the computer
  - *e.g.*, can it parallelise some operations?
- So we need a "universal" model computer to create formal arguments about what is possible
- Turing created one, way back when we were just inventing computing

Turing Machines

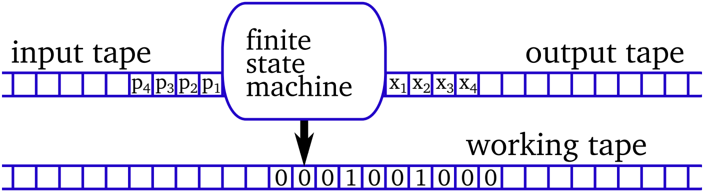
- An abstract model of a computer
- Turns out that all sufficiently complex computing systems are equivalent in the sense that they can compute the same family of functions:
    - computable functions intuitively have a finite program, that completes in a finite number of steps to the result
    - almost all functions we deal with in math are computable (though maybe not efficiently)
    - there are a few that aren't
- Turing machines have a few variants, but simplest has
    - a tape
    - a finite state machine that can write/read from the tape

Simple Turing Machine

- a tape
  - a **tape** is an idealisation of computer memory
  - imagine a strip of paper on which we can write or erase some symbols (often binary 1s and 0s)
  - the tape can be moved back and forth so that the machine can write and read any point on the tape
- a finite state machine that can write/read from each tape
  - n states, plus "halt"
  - transition function has inputs of current state and current tape value
  - transition causes three outputs:
    - ★ can write over the current bit of the tape
    - ★ it can move the tape
    - ★ the state machine's state can change
- running the machine means setting a set of tape values, and a starting state, and then allowing transitions until "halt" is reached

Our Turing Machine

- Ours will be just a little different (but equivalent)



- Its helpful to separate inputs and outputs from working memory
  - input tape (with the input **p** – the *program* – on it)
  - output tape (which we will write the output **x** on)
  - a working tape
  - a finite state machine that can write/read from each tape
- We'll call this a *universal computer*
  - measure complexity by the number of state changes

Section 2

Complexity Nomenclature

Algorithm v Problem complexity

Remember that

- we start with *instances* of a class of problem of given size n
- in general think about solving using a Turing Machine
- we can measure the time of an instance, and often calculate the time of the worst case, when a particular *algorithm* is used to solve it
- we often attribute a *problem* with the complexity of the best algorithm, on the worst instance
- describe complexity with big-O notation, *e.g.,* $O(n^2)$

General problem descriptions

Common types of (general) problem

- decision : does a solution exist?
- search : find a solution.
- counting : how many solutions exist?
- optimisation : find the best solution.

The distinction is arbitrary: *e.g.*, we can solve a decision problem by searching for a solution, but its helpful in thinking about complexity.

General problem descriptions: example 1

Example

decision	:	is n prime?
search	:	factorise n (if it is possible).
counting	:	how many possible factors does n have?
optimisation	:	find the factorisation which has the largest sum of factors.

General problem descriptions: example 2

Example

Given a set of numbers, *e.g.*, $S = \{-7, -3, -2, 5, 8\}$

decision : does some subset of S add to give zero? Yes.

search : find a subset that adds to give zero: $\{-3, -2, 5\}$

counting : how many subsets of S add to give zero? 1?

optimisation : find the subset that adds to zero with the least members. $\{-3, -2, 5\}$

General problem descriptions: example 3 (TSP)

Example

TSP (Travelling Sale-person's Problem) variants:

decision	: is there a path visiting each city with distance less than k?
search	: find a path visiting each city with distance less than k.
counting	: how many paths have distance less than k?
optimisation	: find the shortest path visiting all cities.

General problem descriptions: example 4 (LP)

Example

Linear programming problems:

decision : is $A\mathbf{x} \le \mathbf{b}$ feasible?
Simplex Phase I, or is there something easier?

search : find a feasible solution to $A\mathbf{x} \le \mathbf{b}$. Simplex Phase I

counting : how many vertices does the region defined by $A\mathbf{x} \le \mathbf{b}$ have?
This could be hard?

optimisation : maximise $\mathbf{c}^T\mathbf{x}$ over $A\mathbf{x} \le \mathbf{b}$. Simplex

Decision problems

Definition (Decision problem)

A *decision problem* is a problem whose answer is YES or NO.

- Can be viewed as dividing problem instances in member and non-member instances
- Avoids issues of the output size of the problem
- The alternatives above could be described as *function problems* where a more complex result is the output. It seems a richer class of problem, but can always be recast as decision problems, *e.g.*,

$$a \times b$$

  can be recast as a set of problems "is $a \times b = c$?"
- NB: often, we solve decision problems by searching for a solution! Think of the solution as a *check*.

Tractability

Problems that can be solved in theory (*e.g.*, given large but finite time), but which in practice take too long for their solutions to be useful, are known as *intractable* problems

We can't trivially distinguish the intractable and tractable problems, so we often divide them by their asymptotic performance into

polynomial
: means there is an algorithm which takes time $poly(n)$ for some polynomial $p(n)$ on inputs of length n
  - remember this is the worst case performance
  - write it as $O(n^k)$ for some fixed k
  - polynomial time algorithms are often treated as equivalent to tractable

exponential
: means it takes time *at least* $2^{poly(n)}$
  - grows faster than any polynomial
  - exponential algorithms are assumed to be intractable

Section 3

P v NP

Deterministic and Non-Deterministic Turing Machines

- A *deterministic* Turing machine is just what we described earlier
  - given a particular input, its output is "deterministic"
  - people have built them (almost)
  - standard computers are analogous
- a *non-deterministitc* Turing machine: it can have a set of rules that give more than one action for a given situation.
  - in a given state, input a given symbol, perform both A and B
  - can think of it as
    - getting to try both possibilities
    - being able to guess the correct branch

  Non-deterministic Turing machines don't exist, but are useful for describing algorithm complexity.

P and NP

Definition (P)

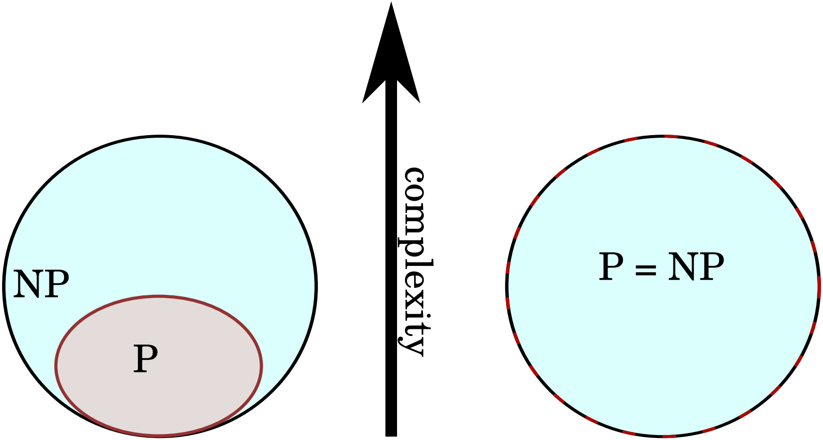
P is the set of *decision* problems that are *Polynomial time*, *i.e.*, they can be solved by a deterministic Turing machine in polynomial time.

Definition (NP)

NP is the set of *decision* problems that are *Non-deterministic Polynomial time*, *i.e.*, they can be solved by a non-deterministic Turing machine in polynomial time.

- NP does NOT mean Non-Polynomial
- It actually includes all polynomial-time decision problems, *i.e.*, $P \subset NP$
- We *don't know* if it has anything else in it
  - Is $P = NP$?
  - Win $\$1,000,000$ if you can answer this

Euler Diagrams



$P \neq NP$ $\qquad$ $P = NP$

NP = Non-deterministic Polynomial time

Ways to think about NP

- NP problems have an efficient (polynomial time) *verifier*
    - computing the decision might be hard
    - but checking a YES decision is easy
    - *e.g.*, is n prime?
        - factorization might require checking all possible factors
        - given two factors p, q its easy to check $p \times q = n$

  assumes the YES result comes with a "proof certificate" (often a solution) which can be checked.

- They can be solved in polynomial time by a non-deterministic Turing machine using the following approach
    1. Guess a solution
    2. Check it

  A non-deterministic Turing machine can make the right guess, so compute time is just the time to check the solution.

NP examples

- All P problems
- Graph isomorphism problem
- integer factorisation
- SAT

SAT

A very general class of decision problems is SAT

Definition (SAT)

A (Boolean) *satisfiability* (SAT) problem has n Boolean variables $x_1, \ldots, x_n$ and a Boolean formula ϕ involving the variables. The question is whether there is an assignment (of TRUE and FALSE) to the variables, such that $\phi(x_1, \ldots, x_n) = TRUE$, *i.e.*, we satisfy the formula.

Example

One variable x_1 and Boolean formula

$$\phi(\mathbf{x}) = x1 \wedge \neg x1$$

where $\wedge =$ AND and $\neg =$ NOT, is *not satisfiable* because

$$\begin{aligned} TRUE\ AND\ NOT\ TRUE &= FALSE \\ FALSE\ AND\ NOT\ FALSE &= FALSE \end{aligned}$$

so there is no value of x_1 that leads to $\phi(x_1) = TRUE$.

SAT

Example

Three variables x_1, x_2 and x_3 and Boolean formula

$$\phi(\mathbf{x}) = (x1 \vee \neg x2) \wedge (\neg x1 \vee x2 \vee x3) \wedge \neg x1$$

where

$$\begin{aligned} \vee &= \text{OR} \\ \wedge &= \text{AND} \\ \neg &= \text{NOT} \end{aligned}$$

is satisfied by $x1 = \mathit{FALSE}$, $x2 = \mathit{FALSE}$, and $x3$ arbitrarily.

We *think* some problems in NP aren't in P

- We certainly know some problems in NP for which we have no polynomial-time algorithm *at present*
  - *e.g.*, SAT
  - so we think these might be harder than P
  - a problem is called *NP-hard* if it is at least as hard as the hardest problem in NP
  - we'll define formally in a moment
- If $P \neq NP$ then NP-hard problems cannot be solved in polynomial time.

We *think* some problems in NP aren't in P

> If $P = NP$, then the world would be a profoundly different place than we usually assume it to be. There would be no special value in "creative leaps," no fundamental gap between solving a problem and recognizing the solution once it's found. Everyone who could appreciate a symphony would be Mozart; everyone who could follow a step-by-step argument would be Gauss; everyone who could recognize a good investment strategy would be Warren Buffett. Its possible to put the point in Darwinian terms: if this is the sort of universe we inhabited, why wouldn't we already have evolved to take advantage of it?
>
> *Scott Aaronson*

http://www.scottaaronson.com/blog/?p=122

NP-hard definitions

Definition

A problem A can be reduced to B if we could solve A using the algorithm that solves B as a subroutine.

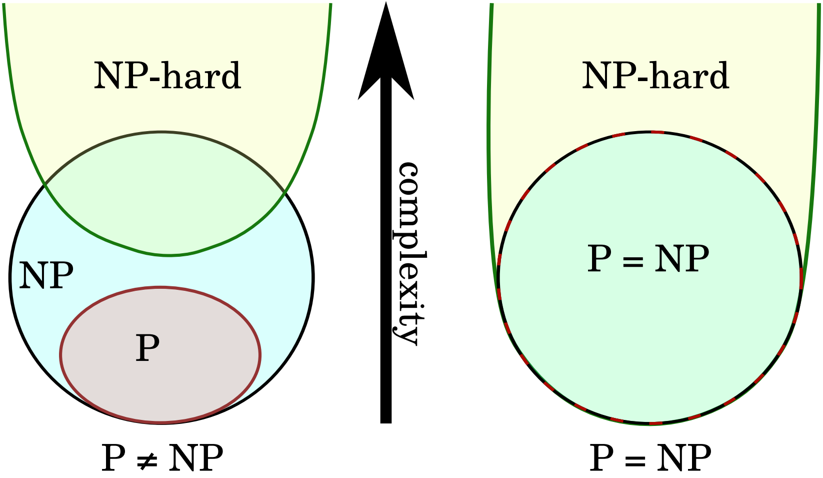
- If we have a polynomial time reduction (that is one that can be done in polynomial time, excluding the time in the subroutine) then we can efficiently convert one problem into the other.
- So A is no more difficult than B.

Definition (NP-hard)

A problem H is NP-hard if every problem L in NP can be *reduced* in polynomial time to H.

- So H is at least as hard as any L in NP.
- Note that an NP-hard problem isn't necessarily in NP!

Euler Diagrams



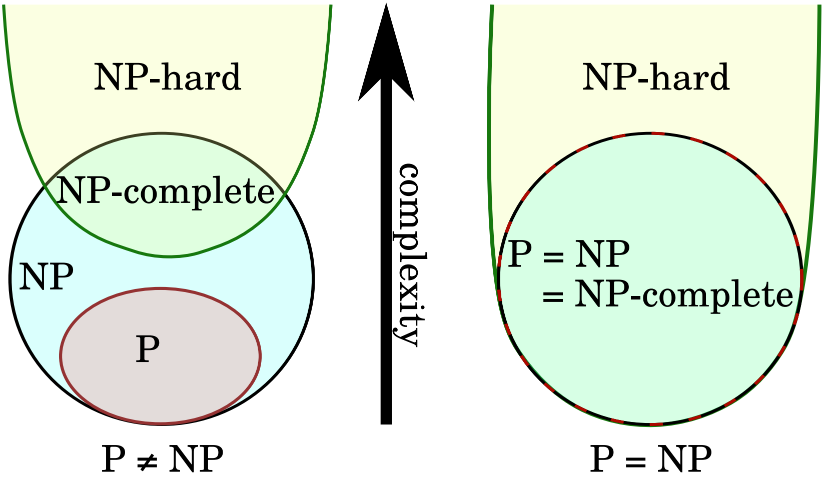
NP-complete

Definition (NP-complete)

A problem that is in NP, and in NP-hard is called NP-complete.

- A problem p in NP is NP-complete if every other problem in NP can be reduced to p in polynomial time.
- A decision problem is NP-complete if it is in NP, and every problem in NP is reducible to it.
- Cook's theorem: the Boolean satisfiability problem (SAT) is NP-complete
  - so many proofs of NP-completeness show SAT can be reduced to the problem

Euler Diagrams



Example NP-complete problems

- SAT (and many variants)
- *Binary linear programming*
- Set covering
- Hamiltonian circuit
- Graph colouring
- Bin-packing and Knapsack
- TSP problem
- Many others ....

NB: decision versions of the above where its not obvious.

Other important chunks

- If a decision problem is NP-complete, then its optimisation version is NP-hard
- Weirdest case I know
  - The graph isomophism problem
    - is graph G_1 isomophic to G_2
    - its in NP
    - its suspected to be neither in P or NP-complete
    - very recently a *quasi-polynomial* time algorithm was found

  call these NPI = NP-intermediate
  - in NP, but not in P or NP-complete
- There are NP-hard problems that are not NP-complete
  - *e.g.*, the halting problem
    - given a program and its input, will it run forever?
    - its undecidable (so not in NP)
    - SAT can be reduced to the halting problem by writing Turing machine program that tries all values

Misconceptions

- NP-complete problems are not the "hardest"
  - they are in NP – some problems aren't!
    - some problems can't even be verified in polynomial time
  - decision problems in Presburger arithmetic can take $O(2^{2^n})$, *i.e.*, double exponential time
- Not all *instances* of NP-complete problems are hard
  - many (even most) instances of some NP-complete problems can be solved in polynomial time
  - complexity refers to worst case
- Problems with an exponential number of possibilities are not all NP-complete
  - counter-example: shortest paths is solvable in $O(n \log n)$ time

Takeaways

- We talked about complexity *classes*
  - P
  - NP
  - NP-complete
  - NP-hard

  we don't know if $P = NP$
- *At the moment*, we can't solve an NP-complete problem in guaranteed polynomial time
  - integer programming is, in general, NP-complete
    - some of these problems are currently intractable
    - but some restricted subsets of integer programming problems might have polynomial time algorithms
    - others might have good approximations
  - in general, though, we are going to have to be a bit more clever when tackling integer programming problems
    - there is no "one-size-fits-all" like the Simplex for LPs

Something to watch

Watch https://www.youtube.com/watch?v=YX40hbAHx3s&feature=youtu.be

Further reading I