Transform Methods & Signal Processing
Class Exercise 4: solutions

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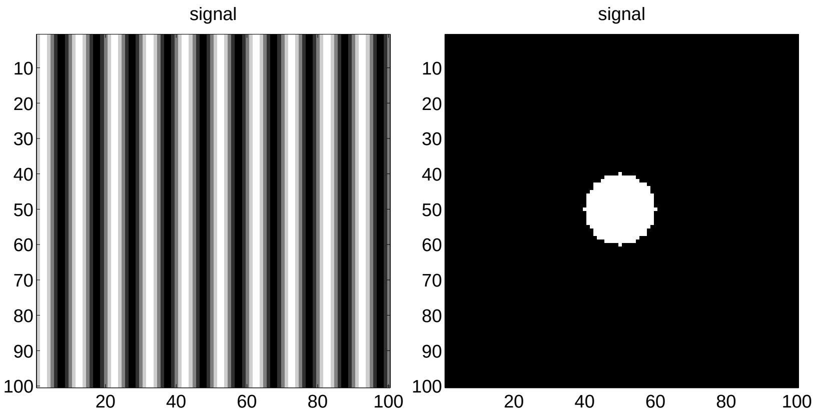
1. [4 marks] Look at the images displayed in figure below (the first is sinusoidal in one direction, and constant in the other, the second is zero outside, and one inside a circle). Describe what the power-spectrum of these images would look like.



Solution: In the first case, the image shows a sinusoid in the x direction, and a constant in the y direction. Note that there are 10 repetitions of the sinusoid, so it is frequency 10. Hence, the power-spectrum will have a delta at the frequency bin corresponding to frequency 10 horizontally, and zero vertically, and the corresponding term for frequency -10. The figure below shows this FT.

In the second case, the function is (approximately) radially symmetric, and so the FT will also have (approximate) radial symmetry. Further, if we took a single slice through the image (say at $y = 50$) we would see a profile that looked like a rectangular pulse. Therefore, we should expect to see the FT of a rectangular pulse (a sinc) when we examine a slice of the image's FT. Therefore the power-spectrum will look like a sinc^2 function rotated around the zero frequency point.

2. [4 marks] Calculate the two-dimensional convolution of $f(x,y) = \delta(x)r(y)$ with $g(x,y) = r(x)\delta(y)$. Hint a 2D convolution is

$$[f * g](x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')g(x-x', y-y')\, dx'\, dy'$$

Derive the Fourier transform of this function.

Solution:

$$[f * g](x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x',y')g(x-x', y-y')\, dx'\, dy'$$

$$= \int_{-\infty}^{\infty} \delta(x')r(x-x')\, dx' \int_{-\infty}^{\infty} r(y')\delta(y-y')\, dy'$$
$$= r(x)r(y)$$

which is just a 2D rectangular pulse. It is a separable function, so we can calculate the FT of the x and y components separately, and as each is a rectangular pulse, the FTs will be sinc functions, i.e.

$$\mathcal{F}\{r(x)r(y)\} = \text{sinc}(x)\text{sinc}(y)$$

Note that the product in space doesn't seem to become a convolution in frequency. However, if we were to write this another way, using the fact that the $r(x)$ is constant with respect to y, and so its FT will be a delta in the y direction, i.e. the FT $\mathcal{F}\{r(x)\} = \mathcal{F}\{r(x)const(y)\} = \text{sinc}(x)\delta(y)$ then we get

$$\begin{aligned}\mathcal{F}\{r(x)r(y)\} &= (\text{sinc}(x)\delta(y)) * (\text{sinc}(y)\delta(x)) \\ &= \text{sinc}(x)\text{sinc}(y)\end{aligned}$$

in much the same way that we calculated the previous convolution.

3. [2 marks] Write down the natural generalization of the Fourier transform to 3 dimensions.

Solution:

$$F(u,v,w) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)e^{-i2\pi(ux+vy+wz)}\, dz\, dy\, dz$$

4*. [5 marks] Give the continuous Fourier transform of the following function

(a) $f(x,y) = \exp\left(-\pi(xcos(\theta) + y\sin(\theta))^2\right)$

Solution: The function is a Gaussian $f(x,y) = \exp\left(-\pi x^2\right)$ rotated through θ degrees. The Fourier transform of a Gaussian, is a Gaussian, e.g. for $f(x,y) = \exp\left(-\pi x^2\right)$, we get FT $F(s,t) = \exp\left(-\pi s^2\right)\delta(t)$, but we must also rotate the Fourier transform in the Fourier domain to get

$$F(s,t) = \exp\left(-\pi(scos(\theta) + t\sin(\theta))^2\right)\delta(-s\sin(\theta) + t\cos(\theta))$$

In more detail

$$\mathcal{F}\{f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\} = \iint_{-\infty}^{\infty} f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)e^{-i2\pi(sx+ty)}\, dx\, dy$$

Take the change of variables (a rotation) $\tilde{x} = x\cos\theta + y\sin\theta$ and $\tilde{y} = -x\sin\theta + y\cos\theta$, then the inverse transform is just a reverse rotation $x = \tilde{x}\cos\theta - \tilde{y}\sin\theta$ and $y = \tilde{x}\sin\theta + \tilde{y}\cos\theta$ and $dx\, dy = d\tilde{x}\, d\tilde{y}$ so we can write the integration as

$$\begin{aligned}\mathcal{F}\{f(x\cos\theta + y\sin\theta, -x\sin\theta + y\cos\theta)\} &= \iint_{-\infty}^{\infty} f(\tilde{x},\tilde{y})e^{-i2\pi(s(\tilde{x}\cos\theta - \tilde{y}\sin\theta) + t(\tilde{x}\sin\theta + \tilde{y}\cos\theta))}\, d\tilde{x}\, d\tilde{y} \\ &= \iint_{-\infty}^{\infty} f(\tilde{x},\tilde{y})e^{-i2\pi(\tilde{x}(s\cos\theta + t\sin\theta) + \tilde{y}(-s\sin\theta + t\cos\theta))}\, d\tilde{x}\, d\tilde{y} \\ &= \iint_{-\infty}^{\infty} f(\tilde{x},\tilde{y})e^{-i2\pi(\tilde{x}\tilde{s} + \tilde{y}\tilde{t})}\, d\tilde{x}\, d\tilde{y}\end{aligned}$$

where $(\tilde{s},\tilde{t})$ are the rotated versions of (s,t), and so $\mathcal{F}\{f(\tilde{x},\tilde{y})\} = F(\tilde{s},\tilde{t})$.