# Transform Methods & Signal Processing

## lecture 02

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# Continuous Fourier Transforms

Fourier's Theorem is not only one of the most beautiful results of modern analysis, but it is said to furnish an indispensable instrument in the treatment of nearly every recondite question in modern physics.

Lord Kelvin



The lecture concerns the definition and properties of the continuous Fourier transform. The course itself will be primarily occupied by the discrete Fourier transform, but the two are directly analogous, and many of the properties are more straight-forward to prove in the continuous case.

# Jean Baptiste Joseph Fourier

March 21, 1768 —
May 16, 1830

- son of a tailor (in Auxerre, France)
  - 12th of 15 children
- involved in the French revolution
  - at one point was arrested
- 1798 Fourier joined Napoleon's army in its invasion of Egypt as scientific adviser
  - helped in archaeological explorations.
- 1802 made Prefect of Grenoble
  - work on heat propagation, and **Fourier series**
- survived Napoleon's arrest, and return, and exile



For more of Fourier's life see
`http://turnbull.dcs.st-and.ac.uk/history/Mathematicians/Fourier.html`



# Fourier series

Can write a periodic function as an (infinite) discrete sum of trigonometric terms, e.g. for period $2\pi$

$$f(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + \sum_{n=1}^{\infty} b_n \sin(nx)$$

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\,dx \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)\,dx \end{aligned}$$



Note that Fourier series are NOT the Fourier transform (though they are related as we shall see below).

# Fourier series as a representation

Fourier series is representing the set of functions with period $2\pi$ in terms of the basis functions $\cos$ and $\sin$, exploiting orthogonality of these functions

$$\int_{-\pi}^{\pi} \cos(nx)\,dx = 0 \qquad \int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(nx)\,dx = 0 \qquad \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx = \pi\delta_{mn}$$

$$\int_{-\pi}^{\pi} \sin(nx)\cos(mx)\,dx = 0$$

$\delta_{mn}$ is the Kronecker delta, $\delta_{mn} = \begin{cases} 1, & \text{if } m = n \\ 0, & \text{otherwise.} \end{cases}$



# Complex Fourier series

Can write Fourier series in complex form

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{inx}$$

$$A_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx$$

NB: $e^{inx} = \cos(nx) + i\sin(nx)$



Example: note the trigonometric formulae

$$\cos\theta\cos\phi = [\cos(\theta-\phi)+\cos(\theta+\phi)]/2$$

So for $n, m \in \mathbb{N}$, $n \neq m$

$$\begin{aligned}\int_{-\pi}^{\pi} \cos(nx)\cos(mx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} \cos((n-m)x)+\cos((n+m)x)\,dx \\ &= \frac{1}{2}\left[-\frac{\sin((n-m)x)}{(n-m)} - \frac{\sin((n+m)x)}{(n+m)}\right]_{-\pi}^{\pi} \\ &= 0\end{aligned}$$

For $n, m \in \mathbb{N}$, $n = m$

$$\begin{aligned}\int_{-\pi}^{\pi} \cos^2(nx)\,dx &= \frac{1}{2}\int_{-\pi}^{\pi} 1+\cos(2nx)\,dx \\ &= \frac{1}{2}\left[x - \frac{\sin(2nx)}{2n}\right]_{-\pi}^{\pi} \\ &= \pi\end{aligned}$$

# Fourier series for other periods

For a function with period $L$, we need to scale the basis functions by $2\pi/L$

$$f(x) = \sum_{n=-\infty}^{\infty} A_n e^{i2\pi nx/L}$$

$$A_n = \frac{1}{L}\int_{-L/2}^{L/2} f(x) e^{-i2\pi nx/L}\, dx$$



It is expected that students of this course will already know how to use Fourier series.

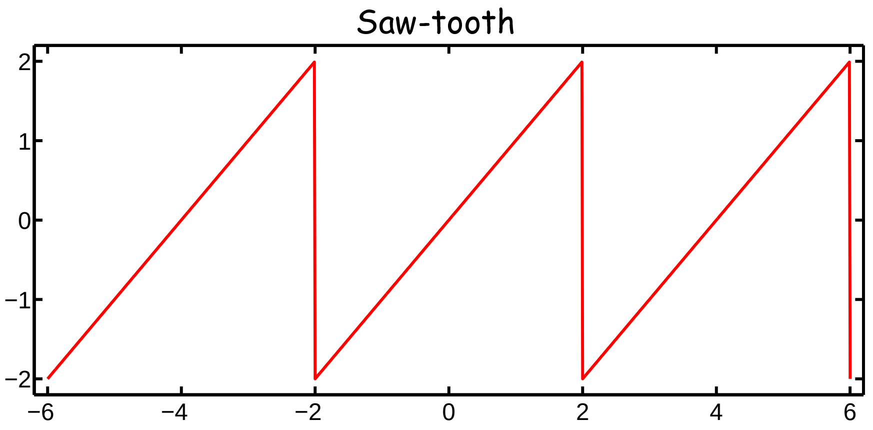
For more information on Fourier series see
`http://mathworld.wolfram.com/FourierSeries.html`

For an instructive java applet see
`http://www.jhu.edu/~signals/fourier2/index.html`



# Example Fourier Series

Fourier series for the saw-tooth $f(x) = x$ for $x = [-L, L]$






For a function with period $L$

$$f(x) = \sum_{i=-\infty}^{\infty} A_n e^{i2\pi nx/L}$$

where for $n \neq 0$

$$A_n = \frac{(-1)^n L}{-2i\pi n}$$

$A_0$ is a special case and $A_0 = 0$. If we take the two components

$$A_n e^{i2\pi nx/L} + A_{-n} e^{-i2\pi nx/L} = \frac{L(-1)^{(n+1)}}{\pi n} \frac{e^{i2\pi nx/L} - e^{-i2\pi nx/L}}{2i} = \frac{L(-1)^{(n+1)} \sin(2\pi nx/L)}{\pi n}$$

we get

$$f(x) = \sum_{n=1}^{\infty} \frac{L(-1)^{(n+1)} \sin(2\pi nx/L)}{\pi n},$$

which is the standard Fourier series representation of the saw-tooth wave form $f(x)$.

## Integral transforms

- An **integral transform** is a transform defined in terms of an integral

$$f(t) \rightarrow \int f(t)g(t,s)dt$$

- Map a function (say of time) to a function of $s$
- $g(\cdot)$ is called the **kernel** of the transform
- notation (several alternatives)
  - $T\{f(t);s\} = \int f(t)g(t,s)\,dt$
  - $F(s) = \int f(t)g(t,s)\,dt$, $H(s) = \int h(t)g(t,s)\,dt$
  - $\mathcal{F}(s) = \int f(t)g(t,s)\,dt$, $\mathcal{H}(s) = \int h(t)g(t,s)\,dt$
  - $\tilde{f}(s) = \int f(t)g(t,s)\,dt$



We will use several of these variants of notation in this course, and you will see other forms, usually as some kind of abbreviation. Part of the goal here is to get you used to some of the different forms of notation you may come across in practise.



## Fourier Transform

Fourier transform $F(s) = \int_{-\infty}^{\infty} f(t)e^{-i2\pi st}\,dt$

Inverse transform $f(t) = \int_{-\infty}^{\infty} F(s)e^{i2\pi st}\,ds$

We are writing function $f(t)$ as a continuous integral of trigonometric functions, weighted by $F(s)$.

- think of as a representation of a function
- sines and cosines are forming a basis
- integral transform with kernel function $g(s,t) = e^{-i2\pi st}$

# Example: FT of a delta function

From the definition of FT

$$\mathcal{F}\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-i2\pi st}\, dt$$

from the definition of a delta

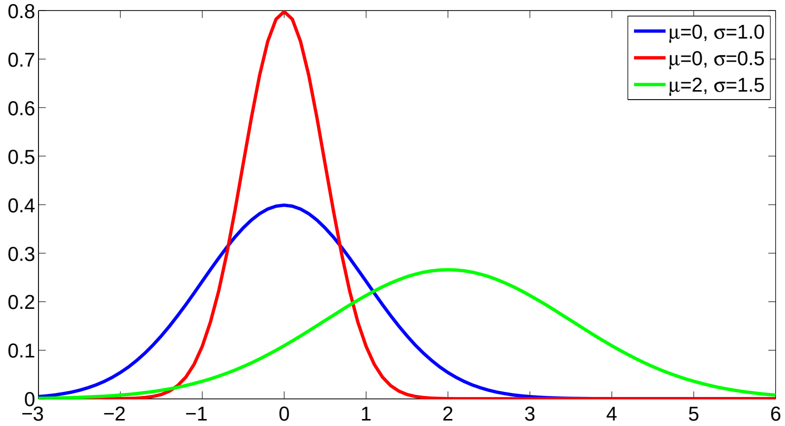
$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)\, dt = f(t_0)$$

$$= e^{-i2\pi st_0}$$



# Gaussian

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(t-\mu)^2/2\sigma^2}$$



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The Gaussian (or normal) probability density is normalized so that its integral is one, i.e.

$$\int_{-\infty}^{\infty} f(x;\mu,\sigma)\, dx = 1.$$

But in general a Gaussian is a function of the form

$$f(t;\mu,a,b) = be^{-a(t-\mu)^2}$$

and we often use this slightly simpler form for signal processing, or even simpler

$$f(t) = e^{-\pi t^2}$$

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# Example: FT of a Gaussian

From the definition of FT

$$
\begin{aligned}
\mathcal{F}\left\{e^{-\pi t^2}\right\} &= \int_{-\infty}^{\infty} e^{-\pi t^2} e^{-i2\pi st}\, dt \\
&= \int_{-\infty}^{\infty} e^{-\pi\left(t^2+i2st\right)}\, dt \\
&= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi(t+is)^2}\, dt \\
&= e^{-\pi s^2} \int_{-\infty}^{\infty} e^{-\pi u^2}\, du \\
&= e^{-\pi s^2}
\end{aligned}
$$



# FT of some simple functions

| Function | Transform |
|---|---|
| $\delta(t)$ | $1$ |
| $\delta(t-t_0)$ | $e^{-i2\pi t_0 s}$ |
| $r(t)$ | $\text{sinc}(s)$ |
| $e^{-\|t\|}$ | $\frac{2}{4\pi^2 s^2+1}$ |
| $e^{-\pi t^2}$ | $e^{-\pi s^2}$ |

| Function | Transform |
|---|---|
| $1$ | $\delta(t)$ |
| $e^{i2\pi s_0 t}$ | $\delta(s-s_0)$ |
| $\text{sinc}(t)$ | $r(s)$ |
| $\frac{2}{4\pi^2 t^2+1}$ | $e^{-\|s\|}$ |
| $e^{-\pi t^2}$ | $e^{-\pi s^2}$ |



See
http://cnyack.homestead.com/files/afourtr/ftgauss.htm

Actually the derivation above is a little fluffy. We really need to recognize that the integral is over a contour in the complex plane, and apply Cauchy's integral theorem: loosely stated, for a contour integral of an analytic function $f(z)$ with continuous partial derivatives around a closed path $\gamma$, then

$$\oint_\gamma f(z)\, dz = 0$$

The integral, after the substitution should really look like

$$\lim_{T\to\infty} \int_{-T+is}^{T+is} e^{-\pi u^2}\, du$$

However, if we form a closed curve around the obvious rectangle then Cauchy gives us

$$\int_{-T+is}^{T+is} e^{-\pi u^2}\, du = \int_{-T}^{T} e^{-\pi u^2}\, du - \int_{T+is}^{T} e^{-\pi u^2}\, du - \int_{-T}^{-T+is} e^{-\pi u^2}\, du$$

We can see that two parts of this go to zero, and the other part gives us the result we need.

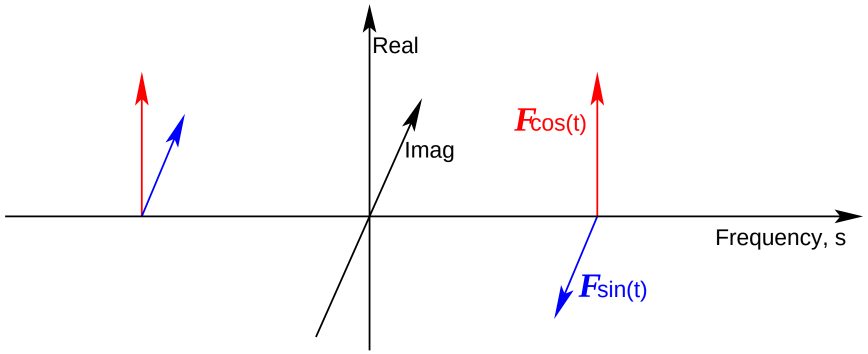
# Deriving a Fourier Transform

We can derive a Fourier transform from scratch, but that can sometimes be hard work. It can be easier to use transforms we already know (and their properties). e.g. exploiting linearity (see later)

$$
\begin{aligned}
\mathcal{F}\{\cos(2\pi s_0 t)\} &= \int_{-\infty}^{\infty} \cos(2\pi s_0 t)\, e^{-i2\pi st}\, dt \\
&= \mathcal{F}\left\{ \frac{1}{2}\left[ e^{-i2\pi s_0 x} + e^{i2\pi s_0 x} \right] \right\} \\
&= \frac{1}{2}\mathcal{F}\left\{ e^{-i2\pi s_0 x} \right\} + \frac{1}{2}\mathcal{F}\left\{ e^{i2\pi s_0 x} \right\} \\
&= \frac{1}{2}\delta(s+s_0) + \frac{1}{2}\delta(s-s_0) \\
\mathcal{F}\{\sin(2\pi s_0 t)\} &= \frac{i}{2}\delta(s+s_0) - \frac{i}{2}\delta(s-s_0)
\end{aligned}
$$



# FTs of sin and cos






Remember that

- $e^{ix} = \cos(x) + i\sin(x)$
- $\cos(x) = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$
- $\sin(x) = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$

Note that Linearity means:

$$\mathcal{F}\{af_1(t) + bf_2(t)\} = a\mathcal{F}\{f_1(t)\} + b\mathcal{F}\{f_2(t)\}$$

and from previous slides

$$\mathcal{F}\left\{e^{i2\pi s_0 t}\right\} = \delta(s - s_0)$$

# The Fourier Transform: definitions

Multiple possible definitions

| Fourier transform | Inverse |
|---|---|
| $F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt$ | $f(t) = \int_{-\infty}^{\infty} F(s) e^{i2\pi st}\, ds$ |
| $F(\omega) = \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$ | $f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$ |
| $F(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$ | $f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(\omega) e^{i\omega t}\, d\omega$ |



# Wave terminology

**Definition:** Amplitude is the extent of a waves oscillation, e.g. a signal $f(t) = A\sin(t)$ has amplitude $f(t)$.

**Definition:** Magnitude is the absolute value of amplitude, e.g. for $f(t) = A\sin(t)$ the amplitude is $|f(t)|$.

**Definition:** Power is the square of magnitude, e.g. for $f(t) = A\sin(t)$ the power is $p(t) = |f(t)|^2$.

**Definition:** RMS Power is the root mean squared power, given by

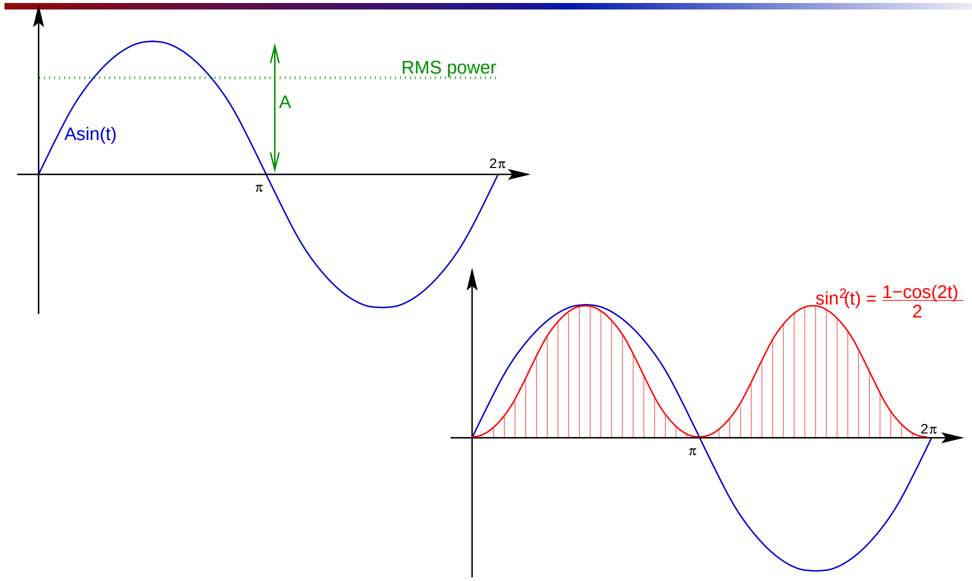
$$m = \sqrt{\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |f(t)|^2\, dt}$$

For $f(t) = A\sin(t)$, the RMS power is $A/\sqrt{2}$, e.g. the RMS power of a sin wave is $0.707$ times the peak value.



We will tend to use the first definition. The second is typically used if we use frequency units of $\omega = 2\pi s$ in radians per second, and we will use this definition in some parts of the the course. The third definition is sometimes prefered because it is symmetric, but we won't use it here.

# RMS power of a sin wave






# RMS power of a sin wave

The sin wave is periodic so we may consider

$$
\begin{aligned}
\frac{1}{2\pi}\int_{-\pi}^{\pi} |A\sin(t)|^2\,dt &= A^2\frac{1}{4\pi}\int_{-\pi}^{\pi} 1-\cos(2t)\,dt \\
&= A^2\frac{1}{2\pi}\left[\int_0^{\pi} 1\,dt - \int_0^{\pi}\cos(2t)\,dt\right] \\
&= A^2\frac{1}{2\pi}[\pi - 0] \\
&= \frac{A^2}{2}
\end{aligned}
$$

To get the RMS power, take the square root, resulting in $A/\sqrt{2}$.



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# Measuring power

- power is a square, so can take wide ranging values.
- use a **log scale** to measure
- ear itself 'hears' logarithmically and humans judge the relative loudness of two sounds by the ratio of their intensities, a logarithmic behavior.
- the typical scale used is **Decibels** (deci- from ten, and Bel from Alexander Graham Bell).
- defined WRT a reference power level $p_{\text{ref}}$

$$\text{power} = 10 \log_{10} \frac{p}{p_{\text{ref}}} dB$$

- $p = m^2$, so we may write power $= 20 \log_{10} \frac{m}{m_{\text{ref}}} dB$
- 3 dB corresponds to a factor of 2 in power



# Decibels and sounds

| Example | Sound Pressure Level (dB) | Sound Intensity (watts/m$^2$) |
|---|---|---|
| Snare drums, played hard at 6 inches | 150 | 1000 |
| 30m from jet aircraft | 140 | 100 |
| Threshold of pain | 130 | 10 |
| Jack hammer | 120 | 1 |
| Fender guitar amplifier, full volume at 10 inches | 110 | 0.1 |
| Subway | 100 | 0.01 |
| | 90 | 0.001 |
| Typical home stereo listening level | 80 | 0.0001 |
| Kerbside of busy road | 70 | 0.00001 |
| Conversational speech at 1 foot away | 60 | $10^{-6}$ |
| Average office noise | 50 | $10^{-7}$ |
| Quiet conversation | 40 | $10^{-8}$ |
| Quiet office | 30 | $10^{-9}$ |
| Quiet living room | 20 | $10^{-10}$ |
| Quiet recording studio | 10 | $10^{-11}$ |
| Threshold of hearing for healthy youths | 0 | $10^{-12}$ |



The eye also perceives subjective brightness as a logarithmic function of light intensity ("Digital Image Processing", Gonzalez and Woods, p.39), but it is less common to use units of dB for light intensity.

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For more information on sound pressure levels see
`http://www.safetyline.wa.gov.au/institute/level2/course18/lecture54/l54_03.asp`

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# Power Spectra

**Definition:** The Power Spectrum of a signal $f(t)$ is $|F(s)|^2$, where $F(s)$ is the Fourier transform,

$$F(s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\, dt$$

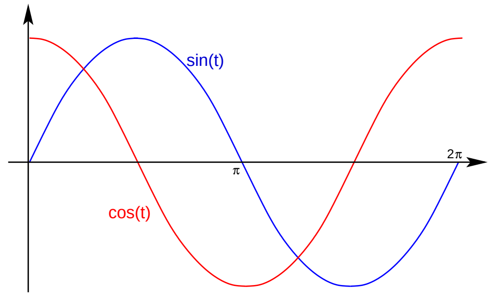
- The power spectrum defines the amount of power at each frequency.
- e.g. $|F(0)|^2$ is referred to as the DC term.
- for real-valued signals the power spectrum is even

$$|F(-s)|^2 = |F(s)|^2$$

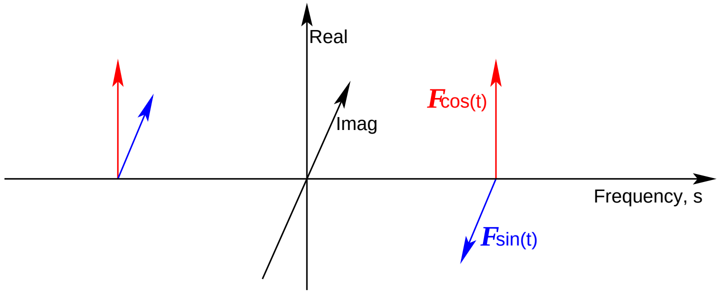
(because the Fourier transform of a real input will be a Hermitian signal).



# Phase




- the sin and cosine functions have the same frequency
- $cos(t) = sin(t + \pi/2)$
- there is a phase change of $\pi/2$



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# Properties of the Fourier transform

| | | | |
|---|---|---|---|
| Linearity: | $af_1(t) + bf_2(t)$ | $\rightarrow$ | $aF_1(s) + bF_2(s)$ |
| Time shift: | $f(t - t_0)$ | $\rightarrow$ | $F(s)e^{-i2\pi s t_0}$ |
| Time scaling: | $f(at)$ | $\rightarrow$ | $\frac{1}{\vert a\vert}F\left(\frac{s}{a}\right)$ |
| Duality: | $F(t)$ | $\rightarrow$ | $f(-s)$ |
| Frequency shift: | $f(t)e^{-i2\pi s_0 t}$ | $\rightarrow$ | $F(s+s_0)$ |
| Convolution: | $f_1(t) * f_2(t)$ | $\rightarrow$ | $F_1(s)F_2(s)$ |
| Differentiation I: | $\frac{d^n}{dt^n}f(t)$ | $\rightarrow$ | $(i2\pi s)^n F(s)$ |
| Differentiation II: | $(-i2\pi t)^n f(t)$ | $\rightarrow$ | $\frac{d^n}{ds^n}F(s)$ |
| Integration: | $\int_{-\infty}^{t} f(s)ds$ | $\rightarrow$ | $\frac{1}{i2\pi s}F(s) + \pi F(0)\delta(s)$ |



# Properties: Linearity

$$\boxed{\mathcal{F}\{af_1(t) + bf_2(t)\} = aF_1(s) + bF_2(s)}$$

$$\begin{aligned}
&\int_{-\infty}^{\infty} [af_1(t) + bf_2(t)]\, e^{-i2\pi st}\, dt \\
&= a\int_{-\infty}^{\infty} f_1(t)e^{-i2\pi st}\, dt + b\int_{-\infty}^{\infty} f_2(t)e^{-i2\pi st}\, dt \\
&= aF_1(s) + bF_2(s)
\end{aligned}$$

- very useful property
- we can use this to derive Fourier transform, e.g. for cos above
- see more on linearity when we discuss filters (lecture 5-6)

# Properties: time shift

$$\mathcal{F}\{f(t-t_0)\} = F(s)e^{-i2\pi s t_0}$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(t-t_0)e^{-i2\pi st}\,dt &= \int_{-\infty}^{\infty} f(t)e^{-i2\pi s(t+t_0)}\,dt \\
&= e^{-i2\pi st_0}\int_{-\infty}^{\infty} f(t)e^{-i2\pi st}\,dt \\
&= e^{-i2\pi st_0}F(s)
\end{aligned}$$

Note $|F(s)e^{-i2\pi st_0}| = |F(s)| \times |e^{-i2\pi st_0}| = |F(s)|$
So the magnitude of the FT is unchanged.

This represents a **phase change**. The higher the frequency, the larger the phase change.

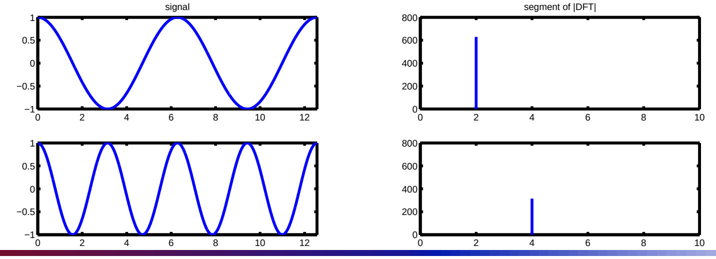


# Properties: Time scaling

$$\mathcal{F}\{f(at)\} = \frac{1}{|a|}F\left(\frac{s}{a}\right)$$

$$\begin{aligned}
\int_{-\infty}^{\infty} f(at)e^{-i2\pi st}\,dt &= \frac{1}{|a|}\int_{-\infty}^{\infty} f(x)e^{-i2\pi (s/a)x}\,dx \\
&= \frac{1}{|a|}F\left(\frac{s}{a}\right)
\end{aligned}$$

$$\begin{aligned}
x &= at \\
dt &= \tfrac{1}{a}\,dx
\end{aligned}$$








This is one of the properties that lead to us equating the results of the Fourier transform with frequency.

# Properties: Duality

$$\boxed{\mathcal{F}\{F(t)\} = f(-s)}$$

Consider the Fourier transform of $F(t)$:

$$\begin{aligned}\int_{-\infty}^{\infty} F(t)e^{-i2\pi st}\,dt &= \int_{-\infty}^{\infty} F(t)e^{i2\pi(-s)t}\,dt, \quad \text{the inverse trans.} \\ &= f(-s)\end{aligned}$$

- the table of Fourier transforms above shows pairs of duals, e.g.

$$\mathcal{F}\{r(t)\} = \mathrm{sinc}(s) \quad \text{and} \quad \mathcal{F}\{\mathrm{sinc}(t)\} = r(t)$$



# Properties: Frequency shift

$$\boxed{\mathcal{F}\left\{f(t)e^{-i2\pi s_0 t}\right\} = F(s+s_0)}$$

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)e^{-i2\pi s_0 t}e^{-i2\pi st}\,dt &= \int_{-\infty}^{\infty} f(t)e^{-i2\pi(s+s_0)t}\,dt \\ &= F(s+s_0)\end{aligned}$$

- used for signal modulation, e.g. FM radio
- simpler using a cos function (see below)



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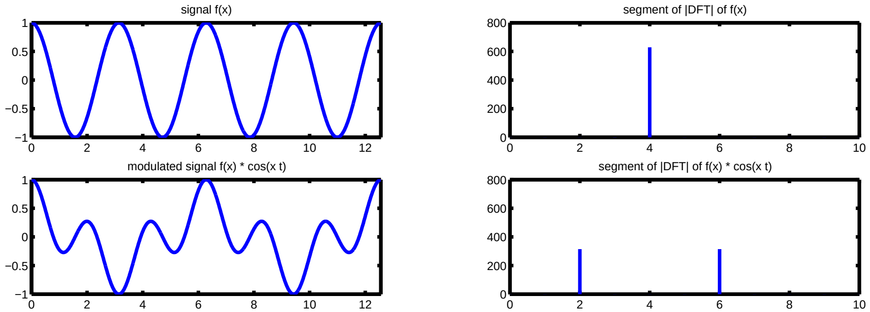
# Properties: Modulation

$$\mathcal{F}\{f(t)\cos(2\pi s_0 t)\} = \frac{1}{2}F(s-s_0) + \frac{1}{2}F(s+s_0)$$

For proof, see freq. shift above noting $\cos x = \frac{1}{2}\left(e^{-ix}+e^{ix}\right)$



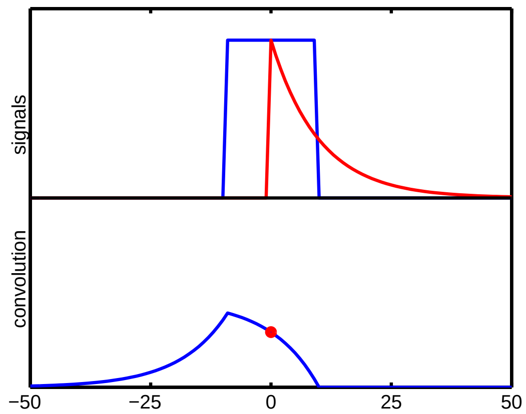

Can use this to generate higher frequency signals, or to demodulate signals.



# Convolutions

What is a convolution?

$$f(t)*g(t) = [f*g](t) = \int_{-\infty}^{\infty} f(u)\,g(t-u)\,du$$






Alternative proof, use the property that $\cos(x)\cos(y) = \frac{1}{2}[\cos(x+y)+\cos(x-y)]$

$$\begin{aligned} y(t) &= \cos(2\pi f_{\text{carrier}} t)\cos(2\pi f t) \\ &= \frac{1}{2}\left[\cos(2\pi[f+f_{\text{carrier}}]t) + \cos(2\pi[f-f_{\text{carrier}}]t)\right] \end{aligned}$$

When modulating a signal, we might then filter out one of these, so that only the single component remains.



Convolution is critical to understanding this course. You must make sure you understand what a convolution is, and how the Fourier transform operates on it.
Java applet that shows the ideas:
`http://www.jhu.edu/~signals/convolve/index.html`

# Properties: Convolution

$$\boxed{\mathcal{F}\{f_1(t)*f_2(t)\} \rightarrow F_1(s)F_2(s)}$$

$$\begin{aligned}
\mathcal{F}\{f(t)*g(t)\} &= \mathcal{F}\left\{\int_{-\infty}^{\infty} f(u)\,g(t-u)\,du\right\} \\
&= \int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(u)\,g(t-u)\,du\right] e^{-i2\pi st}\,dt \\
&= \int_{-\infty}^{\infty} f(u)\int_{-\infty}^{\infty} g(t-u)e^{-i2\pi st}\,dt\,du \\
&= \int_{-\infty}^{\infty} f(u)G(s)e^{-i2\pi su}\,du \\
&= G(s)\int_{-\infty}^{\infty} f(u)e^{-i2\pi su}\,du \\
&= F(s)G(s)
\end{aligned}$$



# Convolution example

Convolution of two rectangular pulses $r(t)$ where

$r(t) = u(t+1/2) - u(t-1/2)$, and $u(t) = \begin{cases} 0, & t<0 \\ 1, & t \geq 0 \end{cases}$

$$\begin{aligned}
r(t)*r(t) &= \int_{-\infty}^{\infty} r(s)\,r(t-s)\,ds \\
&= \begin{cases} 0, & \text{if } t < -1 \\ \int_{-1/2}^{1/2+t} r(s)\,r(t-s)\,ds, & \text{if } -1 \leq t \leq 0 \\ \int_{t-1/2}^{1/2} r(s)\,r(t-s)\,ds, & \text{if } 0 \leq t \leq 1 \\ 0, & \text{if } t > -1 \end{cases}
\end{aligned}$$



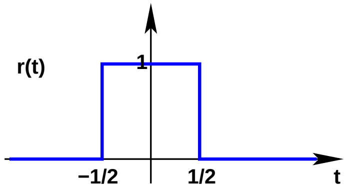
Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.33/54

Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.34/54

# Convolution example

For $-1 \le t \le 0$, the convolution $r(t) * r(t)$ is

$$\begin{aligned} r(t) * r(t) &= \int_{-1/2}^{1/2+t} r(s)\, r(t-s)\, ds, \\ &= \int_{-1/2}^{1/2+t} 1\, ds, \\ &= [t]_{-1/2}^{1/2+t} \\ &= 1/2 + t - -1/2 \\ &= 1 + t, \end{aligned}$$




Similarly for $0 \le t \le 1$, we get $r(t) * r(t) = 1 - t$



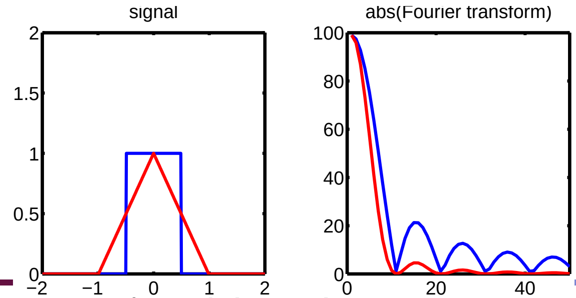
# Convolution example

Result is a Triangular pulse

$$r(t) * r(t) = \begin{cases} 0, & \text{if } t < -1 \\ 1+t, & \text{if } -1 \le t \le 0 \\ 1-t, & \text{if } 0 \le t \le 1 \\ 0, & \text{if } t > -1 \end{cases}$$

$\mathcal{F}\{r(t)\} = \mathrm{sinc}(s)$ hence from the convolution theorem

$\mathcal{F}\{r(t) * r(t)\} = \mathrm{sinc}^2(s)$

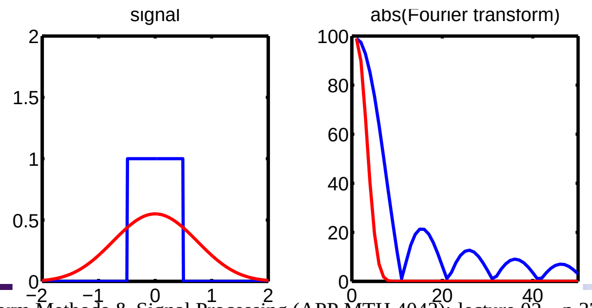


# Limiting convolutions

$$\mathcal{F}\{r(t)\} = \text{sinc}(s) \Rightarrow \mathcal{F}\{r(t) * r(t) * \cdots * r(t)\} = \text{sinc}^n(s)$$

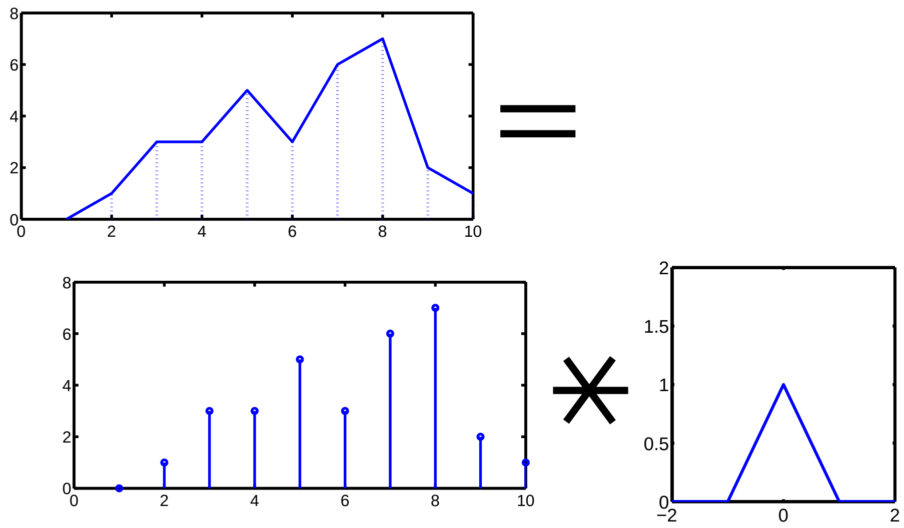
- $n$ convolutions of a rectangular pulse produces a function with FT given by $\text{sinc}^n(x)$, which tends to a Gaussian as $n \to \infty$.
- The inverse FT of a Gaussian is also a Gaussian so the limit of $r(t) * r(t) * \cdots * r(t)$ is a Gaussian pulse.

5 convolutions






# Convolution example: interpolation





Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.37/54

Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.38/54

# Convolution example: interpolation

Fourier transformation of a piecewise linear function

$$f(t) = \left[ \sum_{i=1}^{n} f_i \delta(t - t_i) \right] * r(t) * r(t)$$

is

$$F(s) = \left[ \sum_{i=1}^{n} f_i e^{-i2\pi s t_i} \right] \mathrm{sinc}^2(s)$$



# Properties: Diff. $\frac{d^n}{dt^n} f(t) \rightarrow (i2\pi s)^n F(s)$

$$\boxed{\mathcal{F}\left\{ \frac{d^n}{dt^n} f(t) \right\} = (i2\pi s)^n F(s)}$$

$$\begin{aligned}
\mathcal{F}\left\{ \frac{d}{dt} f(t) \right\} &= \int_{-\infty}^{\infty} \frac{df}{dt} e^{-i2\pi s t} \, dt \\
&= \int_{-\infty}^{\infty} \lim_{\Delta t \rightarrow 0} \frac{f(t+\Delta t) - f(t)}{\Delta t} e^{-i2\pi s t} \, dt \\
&= \lim_{\Delta t \rightarrow 0} \frac{1}{\Delta t} \left[ \int_{-\infty}^{\infty} f(t+\Delta t) e^{-i2\pi s t} \, dt - \int_{-\infty}^{\infty} f(t) e^{-i2\pi s t} \, dt \right] \\
&= \lim_{\Delta t \rightarrow 0} \frac{F(s) e^{i2\pi s \Delta t} - F(s)}{\Delta t} = F(s) \lim_{\Delta t \rightarrow 0} \frac{e^{i2\pi s \Delta t} - e^{i2\pi s 0}}{\Delta t} \\
&= F(s) \left. \frac{d}{dt} e^{i2\pi s t} \right|_{t=0} = i2\pi s F(s)
\end{aligned}$$

and repeat (induction) for higher powers.



Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.39/54

Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.40/54

# Properties: Differentiation II

$$\boxed{\mathcal{F}\{(-i2\pi t)^n f(t)\} = \frac{d^n}{ds^n}F(s)}$$

Similar to previous result,
but with respect to inverse Fourier transform.



# Example: FT of a Gaussian

Another proof of the FT $G(s)$ of a Gaussian $g(t) = e^{-\pi t^2}$.
Note that

$$g'(t) = -2\pi t g(t)$$

From the differentiation property

$$\mathcal{F}\{g'(t)\} = i2\pi s G(s)$$

From the dual differentiation property

$$\begin{aligned} \mathcal{F}\{-i2\pi t g(t)\} &= G'(s) \\ i\mathcal{F}\{g'(t)\} &= G'(s) \\ -2\pi s G(s) &= G'(s) \end{aligned}$$

Standard DE solutions give $G(s) = Ae^{-\pi s^2}$, and the constant $A = 1$ can be derived from the $s = 0$ term.





The $s = 0$ term is just the integral of the Gaussian $g(t)$, and this can be proved to be 1, e.g. see
`http://en.wikipedia.org/wiki/Gaussian_integral`

# Some useful rules for FTs

$$F(-s) = \int_{-\infty}^{\infty} f(t) e^{-i2\pi(-s)t}\, dt = \int_{-\infty}^{\infty} f(-t) e^{-i2\pi st}\, dt$$

Evenness/Oddness of $F(s)$ is related to the properties of $f(t)$.

- even function ⇔ even transform
- odd function ⇔ odd transform



# Some useful rules for FTs

$$F^*(s) = \int_{-\infty}^{\infty} f^*(t) e^{i2\pi st}\, dt = \int_{-\infty}^{\infty} f^*(t) e^{-i2\pi(-s)t}\, dt = \int_{-\infty}^{\infty} f^*(-t) e^{-i2\pi st}\, dt$$

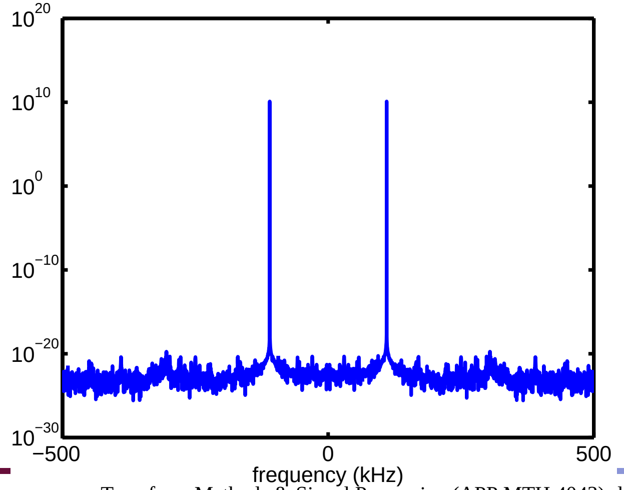
- real even function ⇔ real even transform
- real odd function ⇔ imaginary odd transform



# Some useful rules for FTs

real even function ⇔ real even transform

Magnitude of Fourier transform of a cosine function.



# Properties: Existence

Sufficient conditions

- $\int_{-\infty}^{\infty} |f(t)|\,dt$ exists
- There are a finite number of discontinuities in $f(\cdot)$
- $f(\cdot)$ has bounded variation

The Fourier transform exists for physical signals:
Some conditions above may be technically violated, e.g.

- DC current.
- infinite sin wave
- $\delta(x)$

For first two, can multiply by term like $e^{-ax^2}$, with small $a > 0$ to make integrals exist.



# Properties: Invertible

If the conditions for existence are satisfied.

$$f(t) = \mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\}$$

$$f(t) = \int_{-\infty}^{\infty} \left[ \int_{-\infty}^{\infty} f(t) e^{-i2\pi st}\,dt \right] e^{i2\pi st}\,ds$$

Where $f(t)$ is discontinuous, the equation should be replaced by

$$\frac{1}{2}[f(t^+) + f(t^-)] = \mathcal{F}^{-1}\{\mathcal{F}\{f(t)\}\}$$



Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.46/54

Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.47/54

# Trigonometric basis

- similar to Fourier series: trigonometric functions used as a basis.
- here, can't assume fixed periodicity
- hence must include all sins and cosines
- think of $f(t)$ as containing a mix of periodic functions with different periods
- result is a continuous frequency spectrum



# Measurement of spectra

The (continuous) Fourier transform allows us to examine mathematically the spectra of continuous functions, but is rarely useful in analyzing real signals. However, in some cases we can observe the spectra of real signals directly.



Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.48/54

Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.49/54

# Measurement of Spectra

- how can we use the Fourier transform in practice?
- real signals are effectively continuous
  - sound waves are made of atoms
  - EM waves are made of photons
- how can we analyze frequencies?
  - we don't have an analytic function
  - we can't do the math directly



# Measurement of Spectra

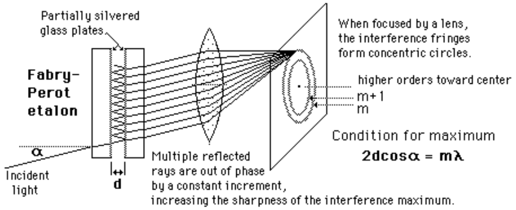
We can measure spectra directly in some cases

- radio frequencies, use a spectrum analyzer
- old ones are analogue
- think of as a bank of filters for each frequency
  - make copies of the signal
  - filter each copy for a particular frequency component
  - one filter per component you want to see

# Measurement of Spectra

We can measure spectra directly in some cases

- light (can use massively parallel analogue devices)
  - prism
  - diffraction grating (a CD)
  - Fabry-Perot interferometer





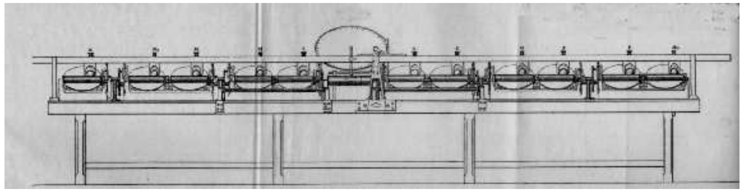

# Measurement of Spectra

We can measure spectra directly in some cases

- tides, "The Harmonic Analyzer" Kelvin, analogues computation of coefficients of

$$A + B\sin t + C\cos t + D\sin 2t + E\cos 2t$$

The tidal gauge, tidal harmonic analyzer, and tide predictor, in Kelvin, Mathematical and Physical Papers (Volume VI), Cambridge 1911, pp 272-305.



http://www.math.sunysb.edu/~tony/tides/analysis.html



Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.52/54

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# Measurement of Spectra

- The tidal gauge illustrates a point
- analogue devices
  - are hard to build
  - have limited resolution
  - are inflexible
- digital devices are often better
  - cheaper
  - more flexible
- we need to consider transforms of digital data
  - that's exactly what we'll do in the next lecture



Transform Methods & Signal Processing (APP MTH 4043): lecture 02 – p.54/54