# Variational Methods and Optimal Control
# Class Exercise 6 solutions

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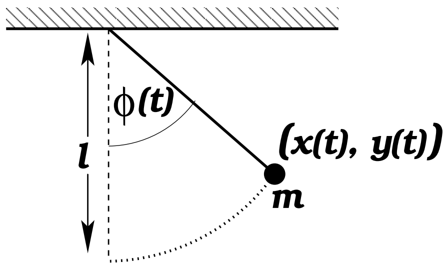
**1**: **Conservation laws:** use Neother's theorem to relate the symmetries of the pendulum to the conservation laws that apply to the system. More specifically, consider the system as follows:




Kinetic energy

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}ml^2\dot{\phi}^2$$

Potential energy

$$V = mg(l-y) = mgl(1-\cos\phi)$$

The Lagrangian is

$$L(\phi, \dot{\phi}) = \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi),$$

and the action integral is

$$F\{\phi\} = \int_{t_0}^{t_1} \left( \frac{1}{2}ml^2\dot{\phi}^2 - mgl(1-\cos\phi) \right) dt.$$

Determine whether the Lagrangian has translation (in space or time) or rotation invariance, and thence determine the conservation laws that apply.

**Solution:**

- The system clearly *does not* possess translational symmetry in the $y$ direction (as there is a $y$ term in the potential energy). There is no explicit $x$ term, but there is an implicit constraint that $x^2 + y^2 = l^2$, and so the system does not possess $x$ translation symmetry either, and hence momentum is *not* conserved.
- The system *does* possess time invariance, and so energy *is* conserved.
- The system *does not* possess rotational invariance (see the $\cos\phi$ term in the functional), and so angular momentum *is not* conserved.

**2.** **Broken extremals:** Minimize the functional

$$F\{x\} = \int_0^2 (\dot{x}+1)^2 \dot{x}^2 \, dt$$

subject to the end-point conditions that $x(0) = 1$ and $x(2) = 0$. *[Hint: consider the possibility of broken extremals.]*

**Solution:** The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dt}\frac{\partial f}{\partial \dot{x}} - \frac{\partial f}{\partial x} &= 0 \\ \frac{d}{dt}\left[ 2(\dot{x}+1)\dot{x}^2 + 2(\dot{x}+1)^2\dot{x} \right] &= 0 \end{aligned}$$





$$\begin{aligned} 2(\dot{x}+1)\dot{x}^2 + 2(\dot{x}+1)^2\dot{x} &= const \\ 2\dot{x}^3 + 2\dot{x}^2 + 2\dot{x}^3 + 4\dot{x}^2 + 2\dot{x} &= const \\ 4\dot{x}^3 + 6\dot{x}^2 + 2\dot{x} &= const \end{aligned}$$

So $\dot{x} = const$, and therefore the solution is $x = at + b$, i.e., the solutions to the Euler-Lagrange equations are straight lines. Naively, we just choose the straight line from $x(0)$ to $x(2)$ to minimize the functional, i.e.,

$$x = 1 - \frac{1}{2}t,$$

which has $\dot{x} = -1/2$, and hence,

$$F\{x\} = \int_0^2 (\dot{x}+1)^2\dot{x}^2 \, dt = \int_0^2 \frac{1}{16} \, dt = \frac{1}{8}.$$

which may seem small, but is actually only a local minimum.

Consider a function with a potential corner at $t = t_c$, and assume slopes of the line are $\dot{x}(t_c^-) = a_1$, and $\dot{x}(t_c^-) = a_2$ on either side of the corner. The Erdman-Weierstrass corner conditions are

$$\begin{aligned} \left.\frac{\partial f}{\partial \dot{x}}\right|_{t_c^-} &= \left.\frac{\partial f}{\partial \dot{x}}\right|_{t_c^+} \\ 4a_1^3 + 6a_1^2 + 2a_1 &= 4a_2^3 + 6a_2^2 + 2a_2 \\ a_1(4a_1^2 + 6a_1 + 2) &= a_2(4a_2^2 + 6a_2 + 2) \\ 2a_1(a_1+1)(2a_1+1) &= 2a_2(a_2+1)(2a_2+1) \qquad (1) \end{aligned}$$

and

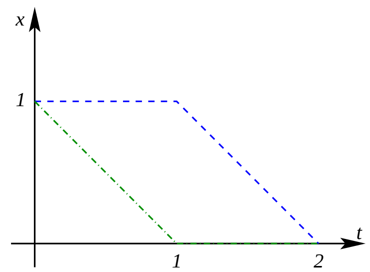
$$\begin{aligned} H|_{t_c^-} &= H|_{t_c^+} \\ \left.\dot{x}\frac{\partial f}{\partial \dot{x}} - f\right|_{t_c^-} &= \left.\dot{x}\frac{\partial f}{\partial \dot{x}} - f\right|_{t_c^+} \\ (4a_1^3 + 6a_1^2 + 2a_1)a_1 - (a_1^4 + 2a_1^3 + a_1^2) &= (4a_2^3 + 6a_2^2 + 2a_2)a_2 - (a_2^4 + 2a_2^3 + a_2^2) \\ a_1^2(3a_1^2 + 4a_1 + 1) &= a_2^2(3a_2^2 + 4a_2 + 1) \qquad (2) \\ a_1^2(3a_1+1)(a_1+1) &= a_2^2(3a_2+1)(a_2+1) \qquad (3) \end{aligned}$$

We can satisfy (3) and (1) by taking $a_i = 0$ or $-1$. If we consider these as possible extremals we immediately note that when $a = 0$ we get $\dot{x} = 0$, and when $a = -1$ we get $1 + \dot{x} = 0$, and so

$$F\{x\} = \int_0^2 (\dot{x}+1)^2\dot{x}^2 \, dt = 0,$$

and given that the squared terms cannot be negative, this is the minimal of the functional.

The extremal is made of straight sections with slope zero, or -1, there being two equally good solutions matching the end-point with one corner, as shown in the figure. If we allow additional corners, then there are many more possibilities.



**3. Optimal control:** Express the following in a form of an optimal control problem to which the Pontryagin Maximum Principle can be applied:

(a) Minimize

$$F\{x\} = \int_0^{10} x^2 \, dt$$

subject to

$$|\ddot{x}| \leq 1, \text{ and } x(0) = 1$$

(b) Minimize $T$ subject to

$$\int_0^T \ddot{x}^2 \, dt = 4$$

and

$$x(0) = 1, \text{ and } \dot{x}(0) = 1, \text{ and } \dot{x}(T) = -2$$

**Solutions**

(a) The constraint $|\ddot{x}| \leq 1$ is not in a suitable form. We need to first write it as a 1st order DE. Start by writing the equivalent constraint

$$\ddot{x}^2 \leq 1$$

and then add a slack variable to create an equation and we get

$$\ddot{x}^2 + \alpha^2 = 1$$

This is a second order DE, and we need to rewrite in terms of first order DEs, so make the substitution

$$\begin{aligned} x_1 &= x \\ x_2 &= \dot{x} \end{aligned}$$

and then we get the equations

$$\begin{aligned} \dot{x_1} &= x_2 \\ \dot{x_2} &= \pm\sqrt{1-\alpha^2} \end{aligned}$$

The functional also needs to be rewritten as

$$F\{x_1, x_2\} = \int_0^{10} x_1^2 \, dt$$

and likewise the end-point constraint.

(b) This is a time minimization problem so we seek to minimize the integral of $\int_0^T 1 \, dt$. Including a Lagrange multiplier for the isoperimetric constraint $\int_0^T \ddot{x}^2 \, dt = 4$ we need to minimize

$$F\{x\} = \int_0^T 1 - \lambda \ddot{x}^2 \, dt$$

Again, this involves second order terms so we use the same change of co-ordinates to $(x_1, x_2)$ as above to write this as minmize

$$F\{x_1, x_2\} = \int_0^T 1 - \lambda \dot{x_2}^2 \, dt$$

subject to

$$x_1(0) = 1, \text{ and } x_2(0) = 1, \text{ and } x_2(T) = -2$$





**4. Optimal control:** A person is considering a lifetime plan of investment and expenditure. With initial savings $S$ and no other income other than from an investment with a fixed interest rate $\alpha > 0$, this investor's capital weath at time $t$ is $x(t)$ and is governed by

$$\dot{x} = \alpha x - r$$

where $r = r(t)$ is the investors rate of expenditure. The immediate enjoyment due to expenditure at rate $r(t)$ results in utility $U(r)$, which we will take to be $U(r) = \sqrt{r}$. Future enjoyment at time $t$ is discounted by $e^{-\beta t}$. Thus our investor wishes to maximize

$$J\{r\} = \int_0^T e^{-\beta t} U(r) \, dt$$

subject to $\dot{x} = \alpha x - r$, and the initial condition $x(0) = 1$. Also, at the final time, any remaining capital is wasted, so let $x(T) = 0$. There are additional implicit constraints: we cannot borrow, so capital cannot become negative, and we cannot expend a negative amount, so $r(t) \geq 0$ for all $t$.

Use the Pontryagin Maximum Principle to find the optimal expenditure strategy $r(t)$.

**Solutions:** Given a minimization problem in the form: minimize functional

$$F = \int_{t_0}^{t_1} f_0\,(t, \mathbf{x}, \mathbf{u}) \, dt,$$

subject to constraints $\dot{\mathbf{x}} = \mathbf{f}(t, \mathbf{x}, \mathbf{u})$, or more fully,

$$\dot{x}_i = f_i(t, \mathbf{x}, \mathbf{u}).$$

The Pontryagin Maximum Principle (PMP) states that for $\mathbf{u}(t)$, an admissible control vector that transfers $(t_0, \mathbf{x}_0)$ to a target $(t_1, \mathbf{x}(t_1))$ and trajectory $\mathbf{x}(t)$ corresponding to $\mathbf{u}(t)$, in order that $\mathbf{u}(t)$ be optimal, it is necessary that there exists $\mathbf{p}(t) = (p_1(t), p_2(t), \ldots, p_n(t))$ and a constant scalar $p_0$ such that

- $\mathbf{p}$ and $\mathbf{x}$ are the solution to the canonical system

$$\dot{\mathbf{x}} = \frac{\partial H}{\partial \mathbf{p}} \quad \text{and} \quad \dot{\mathbf{p}} = -\frac{\partial H}{\partial \mathbf{x}}$$

- where the Hamiltonian is $H = \sum_{i=0}^n p_i f_i$ with $p_0 = -1$
- $H(\mathbf{x}, \mathbf{u}, \mathbf{p}, t) \geq H(\mathbf{x}, \hat{\mathbf{u}}, \mathbf{p}, t)$ for all alternate controls $\hat{\mathbf{u}}$
- all boundary conditions are satisfied

The state variable here is $x$, and the control variable is $r$. The functions of interest here (noting that the problem is a maximization problem, and the PMP is written in terms of minimization) are

$$\begin{aligned} f_0\,(x, r) &= -e^{-\beta t} r^{1/2} \\ f_1\,(x, r) &= \alpha x - r \end{aligned}$$

so the Hamiltonian is

$$H = p(\alpha x - r) + e^{-\beta t} r^{1/2}.$$

The canonical DEs are

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p} = \alpha x - r, \quad \text{the state equation} \\ \dot{p} &= -\frac{\partial H}{\partial x} = -\alpha p. \end{aligned}$$

The second equation gives

$$p = A e^{-\alpha t}.$$

Maximizing $H$ with respect to $r$, means we take

$$\frac{\partial H}{\partial r} = -p + \frac{1}{2} e^{-\beta t} r^{-1/2} = 0.$$

So

$$\begin{aligned} r^{1/2} &= \frac{e^{-\beta t}}{2p} \\ &= \frac{e^{(\alpha-\beta)t}}{2A} \\ r &= \frac{e^{2(\alpha-\beta)t}}{4A^2} \end{aligned}$$

We can then substitute this into the state equation to get $x$, i.e.,

$$\begin{aligned} \dot{x} &= \alpha x - r \\ &= \alpha x - \frac{e^{2(\alpha-\beta)t}}{4A^2} \\ x &= B e^{\alpha t} - \frac{e^{2(\alpha-\beta)t}}{4A^2(\alpha - 2\beta)} \end{aligned}$$

However, we want $x(0) = 1$ so

$$\begin{aligned} B - \frac{1}{4A^2(\alpha-2\beta)} &= 1 \\ B &= \frac{1 + 4A^2(\alpha - 2\beta)}{4A^2(\alpha-2\beta)} \end{aligned}$$

and we want $x(T) = 0$ so (assuming $\alpha - 2\beta \neq 0$)

$$\begin{aligned} B e^{\alpha T} - \frac{e^{2(\alpha-\beta)T}}{4A^2(\alpha-2\beta)} &= 0 \\ (1 + 4A^2(\alpha - 2\beta)) e^{\alpha T} &= e^{2(\alpha-\beta)T} \\ 4A^2(\alpha - 2\beta) &= e^{(\alpha-2\beta)T} - 1 \\ A^2 &= \frac{e^{(\alpha-2\beta)T} - 1}{4(\alpha - 2\beta)} \end{aligned}$$

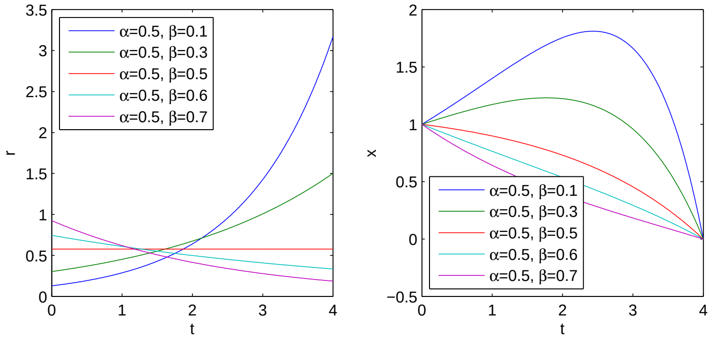
from which we can derive $A$, and thence $B$ is

$$\begin{aligned} B &= \frac{1 + 4A^2(\alpha-2\beta)}{4A^2(\alpha - 2\beta)} \\ &= \frac{e^{(\alpha-2\beta)T}}{e^{(\alpha-2\beta)T} - 1} \end{aligned}$$

The figure shows the derived $r$ and $x$ curves.








Note that the objective function can be calculated to give

$$\begin{aligned} J\{r\} &= \int_0^T e^{-\beta t} r^{1/2}\, dt \\ &= \int_0^T e^{-\beta t} \frac{e^{(\alpha-\beta)t}}{2A}\, dt \\ &= \int_0^T \frac{1}{2A} e^{(\alpha-2\beta)t}\, dt \\ &= \frac{e^{(\alpha-2\beta)T} - 1}{2(\alpha-2\beta)A} \end{aligned}$$