Variational Methods & Optimal Control

lecture 30

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Revision.

Fixed end points: lecture 4

Let $F : C^2[x_0, x_1] \to \mathbb{R}$ be a functional of the form

$$F\{y\} = \int_{x_0}^{x_1} f(x, y, y')\, dx,$$

where f has continuous partial derivatives of second order with respect to x, y, and y', and $x_0 < x_1$. Let

$$S = \left\{ y \in C^2[x_0, x_1] \,\middle|\, y(x_0) = y_0 \text{ and } y(x_1) = y_1 \right\},$$

where y_0 and y_1 are real numbers. If $y \in S$ is an extremal for F, then for all $x \in [x_0, x_1]$

$$\boxed{\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0} \Leftarrow \text{the Euler-Lagrange equation}$$

Special Cases: lectures 4-8

- f depends only on y'
  - e.g., geodesics in the plane
  - always results in straight lines
- f has no explicit dependence on x (autonomous case)
  - e.g., the catenary, brachystochrone, Newton's nosecone
  - use the Hamiltonian (sometimes)
- f has no explicit dependence on y
  - e.g., the geodesic on the sphere
  - $\partial f / \partial y' = const$
- $f = A(x,y)y' + B(x,y)$ (degenerate case)
  - E-L results in identity

Invariance: lecture 8

Extensions: lecture 9-11

Points to remember:

$$F\{y, z\} = \int f(z, y, z, y', z', z'')\, dx$$

dependent variables $y(x)$ and $z(x)$
use multiple E-L equations

independent variable x
if there is more than one, the E-L equation is a partial DE

higher order derivative z''
use Euler-Poisson equation

And we can combine each of the above if more than one case applies.

Numerical methods: lecture 12-13

Constraints: lecture 14-16, 21

- Integral constraints of the form

$$\int g(x, y, y')dx = const$$

  e.g., the Isoperimetric problem.
  - use Lagrange multiplier constant λ
- Holonomic constraints, e.g., $g(x, y) = 0$
  - use Lagrange multiplier function $\lambda(x)$
- Non-holonomic constraints, e.g., $g(x, y, y') = 0$
  - use Lagrange multiplier function $\lambda(x)$
- Inequality constraints, e.g., $y(x) \geq g(x)$
  - either E-L equations, or constraint $y(x) = g(x)$
  - take care at corners, but often $y' = g'$

Free end points: lecture 17-19

- free at both end points

$$\left[p\delta y - H\delta x\right]_{x_0}^{x_1} = 0 \text{ where } p = \frac{\partial f}{\partial y'} \text{ and } H = y'\frac{\partial f}{\partial y'} - f$$

- separable end points: $p\delta y - H\delta x\Big|_{x_i} = 0$
- fixed x, free y, so $\delta x \neq 0$ and $\delta y = 0$ so $H|_{x_i} = 0$
- fixed y, free x, so $\delta x = 0$ and $\delta y \neq 0$ so $p|_{x_i} = 0$
- terminal cost $\phi(t_1, x_1(t_1))$, free $(t_1, x_1(t_1))$

$$\left[\left(\frac{\partial\phi}{\partial x} + p\right)\delta x + \left(\frac{\partial\phi}{\partial t} - H\right)\delta t\right]_{(t_1,x_1)} = 0$$

Free end points: lecture 17-19

- higher order derivatives: $f(x,y,y',y'')$

$$\left.\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right|_{x_0} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y'} - \frac{d}{dx}\frac{\partial f}{\partial y''}\right|_{x_1} = 0$$

$$\left.\frac{\partial f}{\partial y''}\right|_{x_0} = 0 \quad \text{and} \quad \left.\frac{\partial f}{\partial y''}\right|_{x_1} = 0$$

  - first set replace $y(x_i) = y_i$ fixed
    - e.g., a supported beam
  - second set replace $y(x_i) = y_i'$ fixed
    - e.g., a clamped beam

Free end points: lecture 17-19

- multiple dependent variables

$$\sum_{k=1}^{n} p_k \delta q_k - H\delta t = 0 \text{ where } p_k = \frac{\partial L}{\partial \dot{q}_k} \text{ and } H = \sum_{k=1}^{n} \dot{q}_k p_k - L$$

- transversals: end points on curve (x_Γ, y_Γ)

$$\left(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}\right)\cdot(-H, p) = p\frac{dy_\Gamma}{d\xi} - H\frac{dx_\Gamma}{d\xi} = 0$$

- special case $\quad F\{y\} = \int_0^{x_1} K(x,y)\sqrt{1+y'^2}\,dx$

  transversal condition means extremal joins Γ at right angles

Corners: lecture 20

- solve E-L equations
- look for solutions for each end condition
- match up the solutions at a corner x^* so that
  - y is continuous
  - the Weierstrass-Erdman corner conditions hold

  so

$$\begin{aligned} y\big|_{x^*-} &= y\big|_{x^*+} \\ p\Big|_{x^*-} &= p\Big|_{x^*+} \\ H\Big|_{x^*-} &= H\Big|_{x^*+} \end{aligned}$$

  at any 'corner'

Tricks for solving problems

- Exploiting special properties: see special cases
- Hamilton's equations (Canonical Euler-Lagrange equations)
- Hamilton-Jacobi equations
- PMP

Optimal Control: lecture 17, 21-23, 26-28

Conservation laws: lecture 25

Classification: lecture 29