Variational Methods and Optimal Control

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- **Introduction**

  What is the point of this course?

  Example 1: The money pit.

  Example 2: Catenary: shape of a hanging wire.

  Example 3: Brachystochrone: curve of quickest descent.

  Example 4: Dido's problem

- **Revision**

  Extrema of functions of one variable.

- **Extra bits**

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- **Revision, part ii**

  Extrema of functions of multiple variables. Taylor's theorem and the chain rule in N-D. Hessians and classification of extrema.

  Example 1: $f(x_1,x_2) = x_1^2 - x_2^2 + x_1^3$

  Example 2: $f(x_1,x_2) = r - 1/2r^2$, *where* $r^2 = x_1^2 + x_2^2$

  Example 3: $f(x_1,x_2) = x_2^3 - 3x_1^2 x_2$

- **Extra bits**

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- **Revision, part iii**

  Constrained extrema and Lagrange multipliers.

  Example 1: Rectangle of fixed perimeter with maximal area.

  Example 2: Largest area rectangle inscribed in a circle.

  Example 3: Largest area rectangle inscribed in a ellipse.

  Example 4: Maximize $f(x_1,x_2,x_3) = x_1x_2x_3$ *subject to* $x_1x_2 + x_1x_3 + x_2x_3 = 1$, *and* $x_1 + x_2 + x_3 = 3$

  Example 5: Inequality constraint: largest area rectangle inscribed in a unit circle.

  Example 6: Maximize $3x$ *subject to* $x \leq 10$.

- **Revision, part iv**

  Vector space notation.

- **Functionals**

  In CoV we are not maximizing the value of a simple function, we want to find a "curve" that maximizes (or minimizes) a **functional**. Think of functionals as a generalization of a function, except we can think of it as an ∞-dimensional max. problem.

  Example 1: Catenary: the shape of a hanging wire.

  Example 2: Brachystochrone: curve of quickest descent.

  Example 3: Bent elastic beam.

  Example 4: Stimulated plant growth.

  Example 5: Parking a car.

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- **Fixed-end point problems**

  We'll start with the simplest functional maximization problem, and show how to solve by finding the first variation and deriving the Euler-Lagrange equations:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

  Example 1: Catenary: a hanging wire (without a length constraint).

  Example 2: Geodesics in a plane.

- **Special cases**

  Now that we know the Euler-Lagrange (E-L) equations, we can use them directly, but there are some special cases for which the equations simplify, and make our life easier:

  - f depends only on y'
  - f has no explicit dependence on x (autonomous case)
  - f has no explicit dependence on y
  - $f = A(x,y)y' + B(x,y)$ (degenerate case)

- **Special case 1**

  When f depends only on y' the E-L equations simplify to

$$\frac{\partial f}{\partial y'} = const$$

  An example of this is calculating geodesics in the plane (which we all know are straight lines).

  Example 1: Geodesics in a plane.

  Example 2: Fermat's principle and Snell's law

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- **Special case 2**

  When f has no dependence on x we call this an autonomous problem, and we can replace the E-L equations with

$$H(y,y') = y'\frac{\partial f}{\partial y'} - f(y,y') = const$$

  We will see H again later – it often turns out to be a conserved quantity like energy, and so arises naturally in computing the shape of a catenary.

  Example 1: Catenary: the shape of a hanging wire.

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- **Special case 2: autonomous problems continued**

  $$H(y,y') = y'\frac{\partial f}{\partial y'} - f(y,y') = const$$

  We will see H again later – it often turns out to be a conserved quantity like energy, and so arises naturally in computing the shape of the brachystochrone.

  Example 1: Brachystochrone: curve of quickest descent.

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- **Special case 3**

  When f has no explicit dependence on y the E-L equations simplify to give

  $$\frac{\partial f}{\partial y'} = const$$

  An example where we might use this is in calculating geodesics on non-planar objects such as the sphere.

  Example 1: Geodesics on the unit sphere.

  Example 2: Geodesics on other surfaces in $\mathbb{R}^3$.

  Example 3: Geodesics on a cylinder.

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- **Invariance of the E-L equations**

  We side-track here to note that extremals found using the E-L equations don't depend on the coordinate system! This can be very useful – a change of co-ordinates can often simplify a problem dramatically.

  Example 1: Polar (circular) coordinates.

- **Special case 4**

  When $f = A(x,y)y' + B(x,y)$ we call this a degenerate case, because the E-L equations reduce to

  $$\frac{\partial A}{\partial x} - \frac{\partial B}{\partial y} \quad = \quad 0$$

  but we can't necessarily solve these, and when they are true, the functional's value only depends on the end-points, not the actual shape of the curve.

  Example 1: $f(x,y,y') = (x^2 + 3y^2)y' + 2xy$

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- **Extensions**

  Now we consider extensions to the simple E-L equations presented so far:

  – when f includes higher-order derivatives, e.g., $f(x,y,y',y'')$, e.g., the shape of a bent bar.

  – when there are several dependent variables (i.e., y is a vector), e.g., calculating a particles trajectory.

  – when there are several independent variables (i.e., x is a vector), e.g. calculating extremal surface.

- **Extension 1: higher-order derivatives**

  When f includes higher-order derivatives then the E-L equations can be extended, e.g., if the function includes a y'' term, i.e., $f(x,y,y',y'')$, then
$$\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} + \frac{d^2}{dx^2}\frac{\partial f}{\partial y''} = 0$$
  but now we now need extra edge conditions. A simple example we will consider is the shape of a bent bar.

  Example 1: $F\{y\} = \int_0^1 (1+y''^2)\,dx$

  Example 2: $F\{y\} = \int_0^{\pi/2} \left(y''^2 - y^2 + x^2\right)\,dx$

  Example 3: Bent elastic beam.

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- **Extension 2: several dependent variables**

  When there are several dependent variables, i.e., y is a vector, then the E-L equations generalize to give one DE per dependent variable. A simple example is when we calculate the trajectory of a particle in 3D. This section introduces a number of physics ideas/principles: potentials, Lagrangians, Hamilton's principle, Newton's laws of motion, and conservations laws.

  Example 1: $F\{\mathbf{q}\} = \int_0^1 \left(\dot{q}_1^2 + (\dot{q}_2 - 1)^2 + q_1^2 + q_1 q_2\right)\,dt$

  Example 2: Movement of a particle.

  Example 3: Simple pendulum.

  Example 4: Kepler's problem of planetary motion.

  Example 5: Brachystochrone in 3D.

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- **Extension 3: several independent variables**

  When there are several independent variables, e.g., (x,y) and the extremal we wish to find represents, for instance, a surface $z(x,y)$, and f is a function $f(x,y,z(x,y),z_x,z_y)$, then the E-L equation generalizes to give
$$\frac{\partial f}{\partial z} - \frac{\partial}{\partial x}\frac{\partial f}{\partial z_x} - \frac{\partial}{\partial y}\frac{\partial f}{\partial z_y} = 0$$

  Example 1: $F\{z\} = \iint_\Omega 1 + \frac{1}{2}z_x^2 + \frac{1}{2}z_y^2\,dx\,dy$

  Example 2: Minimal area surface.

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- **Numerical Solutions**

  The E-L equations may be hard to solve

  Natural response is to find numerical methods

  - Numerical solution of E-L DE
    - we won't consider these here (see other courses)
  - Euler's finite difference method
  - Ritz (Rayleigh-Ritz)

  * In 2D: Kantorovich's method

- **Euler's finite difference method**

  We can approximate our function (and hence the integral) onto a finite grid. In this case, the problem reduces to a standard multivariable maximization (or minimization) problem, and we find the solution by setting the derivatives to zero. In the limit as the grid gets finer, this approximates the E-L equations.

- **Ritz's method**

  In Ritz's method (called Kantorovich's methods where there is more than one independent variable), we approximate our functions (the extremal in particular) using a family of simple functions. Again we can reduce the problem into a standard multivariable maximization problem, but now we seek coefficients for our approximation.

  Example 1: $F\{y\} = \int_0^1 \left[\frac{1}{2}y'^2 + \frac{1}{2}y^2 - y\right] dx$

  Example 2: Catenary: the shape of a hanging wire.

  Example 3: $F\{z(x,y)\} = \int_{-b}^{b}\int_{-a}^{a} \left(z_x^2 + z_y^2 - 2z\right) dx\,dy$

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- **Constraints**

  We now include additional constraints into the problems:

  – Integral constraints of the form
  $$\int g(x,y,y')dx = const$$
  e.g., the Isoperimetric problem.

  – Holonomic constraints, e.g., $g(x,y) = 0$

  – Non-holonomic constraints, e.g., $g(x,y,y') = 0$

  – We won't consider inequality constraints until later.

- **Integral Constraints**

  Integral constraints are of the form
  $$\int g(x,y,y')dx = const$$
  The standard example of such a problem is Dido's problem. We solve these by introducing the functional analogy of a Lagrange multiplier.

  Example 1: Dido's problem: simplified

  Example 2: Catenary of fixed length

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Example 3: Dido's problem - traditional

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- **Holonomic Constraints**

  Constraints of the form $g(x,y) = 0$, or $g(t,\mathbf{q}) = 0$, which don't involve derivatives of $y(x)$ or $\mathbf{q}$ can also be handled using a Lagrange multiplier technique, but we have to introduce a Lagrange multiplier function $\lambda(x)$, not just a single value λ. Effectively we introduce one Lagrange multiplier at each point where the constraint is enforced.

  Example 1: Geodesics on the sphere.

- **Non-Holonomic Constraints**

  Constraints of the form $g(x,y,y') = 0$, or $g(t,\mathbf{q},\dot{\mathbf{q}}) = 0$, which involve derivatives. They are effectively additional DEs which we need to solve, but we can once again use Lagrange multipliers.

  Example 1: A simple solution for $F\{y\} = \int_a^b f(x,y,y',y'')\,dx$

- **Intro to Optimal Control**

  One way we see non-holonomic constraints is when we consider control problems. In these we seek to control a system described by a DE (the constraint) subject to some input which we can control (optimize).

  Example 1: Stimulated plant growth

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- **Non-fixed end point problems**

  What happens when we don't fix the end-points of an extremal? In this case **natural boundary conditions** are automatically introduced, and these can allow us to solve the E-L equations.

- **Free end points: Fixed x, Free y and/or y'**

  First we'll consider what happens when we allow y or y' to vary at the end-points, but we still keep the x values of the end-points fixed at x_0 and x_1.

  Example 1: Freely supported elastic beam.

  Example 2: Elastic beam fixed at one end point.

- **Intro to Optimal Control (part II)**

  Often in optimal control problems we may specify the initial state, but not the final state. However, there may be a cost associated with the final state, and we include this in the functional to be minimized (or maximized). We call this a terminal cost.

  Example 1: Stimulated plant growth with a free end-point.

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- **Free end points: Free x, y and y'**

  We now allow x to vary as well, although we may apply some condition on the relationship between x and y, for instance that the end point must lie on a curve. In these cases we often rename our extremals, and call them **transversals**.

  Example 1: Shortest-path between two curves.

  Example 2: Orbit transfer problem.

- **Transversals**

  When we consider an extremal joining a curve to a point (or two curves) then we often call the extremal a transversal. The previous condition simplifies in many such cases, for instance, in many situations we look for a transversal that joins the proscribed curve at right angles.

  Example 1: Shortest path from the origin to a curve.

  Example 2: Generalized shortest path between two curves.

  Example 3: Shape of a wire hanging between two curves.

  Example 4: Curve of fastest descent from a point to line.

  Example 5: Shortest-path from a point to a surface.

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- **Broken Extremals**

  Until now we have required that extremal curves have at least two well-defined derivatives. Obviously this is not always true (see for instance Snell's law). In this lecture we consider the alternatives.

  Example 1: $F\{y\} = \int_{-1}^{1} y^2(1-y')^2\,dx$

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- **Inequality Constraints and Optimal Control**

  Earlier we didn't consider inequalities as constraints, but these are needed particularly in control. For instance, often there is a maximum force we can apply to an object. The resulting extremals either (i) satisfy the E-L equations, or (ii) lie along the edge of the constraint. We also get boundary conditions between these two types of regions.

  Example 1: Parking a car.

  Example 2: Shortest-path avoiding an obstacle.

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- **More Optimal Control Examples**

  First we'll cover a bit more terminology, and then some examples.

  Example 1: Dynamic production control.

  Example 2: Optimal economic growth.

  Example 3: Rocket launch.

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- **Hamilton's formulation**

  We've seen the Hamiltonian H earlier on, but haven't explored its full power. Firstly, using H can often result in a simpler approach than solving the E-L equations, e.g., where f has no dependence on x, or where there is more than one dependent variable. More importantly though, this formulation can lead to an understanding of how symmetries in the problem of interest lead to conservation laws. Finally, we will use the Hamiltonian in the Pontryagin Maximum Principle, which we will study soon.

  Example 1: Simple pendulum: Hamilton's formulation.

  Example 2: Simple pendulum: Hamilton-Jacobi approach.

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- **Conservation Laws**

  One of the more exciting things we can derive relates to fundamental physics laws: conservation of energy, momentum, and angular momentum. We can now derive all of these from an underlying principle: Noether's theorem.

  Example 1: Invariance under translations in x is $\equiv$ conservation of H (energy).

  Example 2: Invariance under translations in y is $\equiv$ conservation of p (momentum).

  Example 3: Invariance under rotations is $\equiv$ conservation of angular momentum.

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- **Pontryagin Maximum Principle**

  Modern optimal control theory often starts from the PMP. It is a simple, concise condition for an optimal control.

  Example 1: Stimulated plant growth.

  Example 2: Stimulated plant growth with a free end-point.

  Example 3: Optimal treatment of gout

  Example 4: Lunar lander.

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- **Bang-Bang controllers and other related issues**

  Here we consider more generally what conditions result in a bang-bang controller.

  Example 1: Optimal fish harvesting

  Example 2: Time minimization problem.

  Example 3: Singular control example.

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- **Feedback control systems**

  In all of our previous examples, we solve optimization problem "all at once", i.e., we plan the shape of the curve y to optimize the functional. However, sometimes, we need a control that reacts continuously to perturbations in a system. Such controllers typically utilize feedback.

  Example 1: Liquid level control.

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- **Classification of extrema**

  We have so far typically ignored the issue of classification of extrema, but remember that for simple stationary points we need to look at higher derivatives to see if a stationary point is a maximum, minimum or point of inflection. We need an analogous process for extremal curves as well.