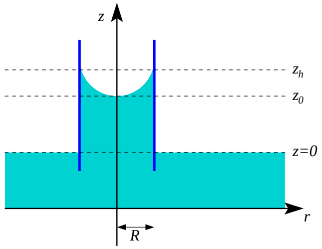
Tutorial 5 Solutions

1. **Capillaries:** Imagine a slender, open-end cylinder dipped into a large water bath. It is well known that the cylinder acts as a capillary tube, and the water will rise up the tube, and moreover that the shape of the surface of the water inside the tube will have a curved form, as shown in the figure.



For convenience, we can assume the water level without any object (and hence at ∞ is $z = 0$, and that the cylinder's center of rotation is the z-axis. Given the radial symmetry of the problem, we will consider it in cylindrical co-ordinates (r, θ, z), and the cylinder will have radius R. We will describe the height of water in the cylinder by $z(r)$, and we denote $z_0 = z(0)$ and $z_h = z(R)$, but note that these are not fixed boundary conditions (the end-points are free).

At equilibrium, the potential energy of this system will be minimized. The potential energy is made up of the following components:

- The gravitational potential:

$$G\{z\} = 2\pi\Delta\rho g \int_0^R r \int_0^z s\,ds\,dr, = 2\pi\Delta\rho g \int_0^R r\frac{z^2}{2}\,dr,$$

where $\Delta\rho$ is the difference between the density of the liquid and air, and g is the gravitational constant.

- The surface energy in the interfaces between the liquid and solid (the cylinder walls), the gas and solid, and the air and liquid, given by

$$S\{z\} = \Delta\gamma\Delta A(\Omega_{SL}) + \gamma_{LG}\Delta A(\Omega_{LG}) = 2\pi R\Delta\gamma z_h + 2\pi\gamma_{LG}\int_0^R r\sqrt{1+z'^2} - r\,dr,$$

where

  – The constant parameters γ_{SG}, γ_{SL} and γ_{LG} are the respective parameters determining the strength of affiliation or attraction between these three components. We can think of a parameter γ as the tension on the surface with units corresponding to force along a line of unit length. We take $\Delta\gamma = \gamma_{SL} - \gamma_{SG}$.

  – Ω_{SG}, Ω_{SL} and Ω_{LG} are the surfaces between the respective phases.
  – $A(\cdot)$ denotes the surface area. $\Delta A(\cdot)$ represents the surface area deformation from the case without the cylinder, for instance, the undeformed surface Ω_{LG} would have area given by a circular disk (radius R, area $\pi R^2 = 2\pi\int_0^R r\,dr$), and the undeformed surface Ω_{LS} would correspond to the liquid in the cylinder at the same height as the water bath.

so the integrals above are trying to minimize the energy resulting from tension in the surfaces due to their deformation from the case without the cylinder. The first integral is the energy in the gas-liquid surface, and the second is the energy resulting in the liquid-solid surface minus the energy from the solid-gas interface it replaces.

Use calculus of variations to determine the height and shape of the water surface inside the cylinder.

Solution: The problem is traditionally solved using the Young-Laplace equation, which we shall derive here using calculus of variations.

Ignoring factors of 2π the functional we wish to minimize is

$$F\{z\} = \Phi(z_h) + \int_0^R f(r, z, z')\,dr,$$

where

$$\begin{aligned} \Phi(z_h) &= R\Delta\gamma z_h, \\ f(r, z, z') &= \Delta\rho g r\frac{z^2}{2} + \gamma_{LG}\left(r\sqrt{1+z'^2} - r\right). \end{aligned}$$

The Euler-Lagrange equations are

$$\begin{aligned} \frac{d}{dr}\frac{\partial f}{\partial z'} - \frac{\partial f}{\partial z} &= 0 \\ \frac{d}{dr}\left(\frac{\gamma_{LG} r z'}{\sqrt{1+z'^2}}\right) &= \Delta\rho g r z \\ \gamma_{LG}\left(\frac{z'}{\sqrt{1+z'^2}} + r\frac{d}{dr}\frac{z'}{\sqrt{1+z'^2}}\right) &= \Delta\rho g r z \\ \gamma_{LG}\left(\frac{z'}{r\sqrt{1+z'^2}} + \frac{d}{dr}\frac{z'}{\sqrt{1+z'^2}}\right) &= \Delta\rho g z \\ \gamma_{LG}\left(\frac{z'}{r\sqrt{1+z'^2}} + \frac{z''}{\sqrt{1+z'^2}} + \frac{-z''z'^2}{(1+z'^2)^{3/2}}\right) &= \Delta\rho g z \\ \gamma_{LG}\left(\frac{z'}{r\sqrt{1+z'^2}} + \frac{z''}{(1+z'^2)^{3/2}}\right) &= \Delta\rho g z \\ \gamma_{LG}\left(\frac{1}{R_1} + \frac{1}{R_2}\right) &= \Delta P z, \end{aligned}$$

where R_1 and R_2 are the two radii of curvature of the surface, and ΔP is the pressure difference. This is conventionally called the Young-Laplace equation.

The cylinder or tube is slender, so we assume R is small, and then the solution to this equation is given by a segment of a sphere, i.e.,

$$(z - z_h)^2 + r^2 = c_r^2.$$

where the sphere has radius c_r so

$$z = z_h - \sqrt{c_r^2 - r^2}.$$

so

$$\begin{aligned} z' &= \frac{r}{\sqrt{c_r^2 - r^2}} \\ \frac{z'}{\sqrt{1+z'^2}} &= \frac{r}{\sqrt{c_r^2 - r^2}} \frac{1}{\sqrt{1 + \frac{r^2}{c_r^2 - r^2}}} \\ &= \frac{r}{c_r} \\ \frac{d}{dr}\frac{z'}{\sqrt{1+z'^2}} &= \frac{1}{c_r} \end{aligned}$$

Note that both radii of curvature are c_r as expected for a spherical section radius c_r. So the Euler-Lagrange equations reduce to

$$\frac{2\gamma_{LG}}{c_r} = \Delta P \left[z_h - \sqrt{c_r^2 - r^2} \right].$$

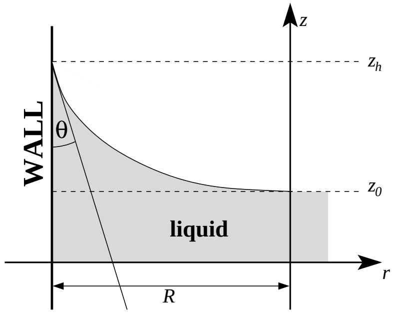
Note that given c_r we can find $z_0 = z(0)$, by taking $r = 0$ and thence

$$z_0 = \frac{2\gamma_{LG}}{\Delta P c_r}. \tag{1}$$

To get c_r we turn to the end-point equations. The end-point z_0 is free so we get natural boundary conditions at this point

$$\begin{aligned} \left.\frac{\partial f}{\partial z'}\right|_{r=0} &= 0 \\ \left.\frac{\gamma_{LG} r z'}{\sqrt{1+z'^2}}\right|_{r=0} &= 0 \end{aligned}$$

which is automatically true.



The end-point z_h is free so we get natural boundary conditions at this point including the terminal cost $\Phi(z_h)$, of the form

$$\begin{aligned} \left.\frac{\partial f}{\partial z'}\right|_{r=R} &= -\frac{\partial \Phi}{\partial z} \\ \frac{\gamma_{LG} R z'}{\sqrt{1+z'^2}} &= -R\Delta\gamma \\ \frac{\gamma_{LG} z'}{\sqrt{1+z'^2}} &= -\Delta\gamma \\ \gamma_{LG}\cos(\theta) &= -\Delta\gamma, \end{aligned}$$

where θ is the angle of contact to the boundary (see figure). This formula is usually arrived at through force balancing arguments, and is often called the Young-Dupre formula.

From this we can derive the radius of the spherical section that forms the surface in the capillary by noting that

$$\cos(\theta) = \frac{R}{c_r},$$

so that

$$c_r = -\frac{\gamma_{LG} R}{\Delta\gamma}.$$

where this only makes sense as long as $c_r > R$. Substituting this into (1) we get

$$z_0 = \frac{2\gamma_{LG}\cos(\theta)}{\Delta P R} = \frac{2\Delta\gamma}{\Delta P R}. \tag{2}$$

which is again a well-known formula, sometimes known as the Jurin rule or Jurin height after James Jurin (1718).

For a glass tube in water, the terms

$$
\begin{aligned}
\gamma_{LG} &= 0.0728 \text{ J/m}^2 \text{ at } 20^o \text{ C},\\
\Delta\gamma &= \gamma_{SL} - \gamma_{SG}.,\\
&= -\gamma_{LG}\cos(\theta), \text{ where for water/glass the contact angle } \theta \text{ is } 20^o,\\
&= -0.06841 \text{ J/m}^2 \text{ at } 20^o \text{ C},\\
\rho_{\text{water}} &= 1000 \text{ kg/m}^3,\\
\rho_{\text{air}} &\simeq 1.225 \text{ kg/m}^3 \text{ at sea level at } 15^o \text{ C},\\
g &= 9.8 \text{ m/s}^2,\\
\Delta P &= \Delta\rho g\\
&= (\rho_{\text{air}} - \rho_{\text{water}})g\\
&\simeq -9,788 \text{ J/m}^2.
\end{aligned}
$$

NB: The unit J stands for a Joule, which is unit of work, and $1J = 1kg \cdot m^2/s^2$.

Given these constants, we can approximate the height of the water column given by (2) by

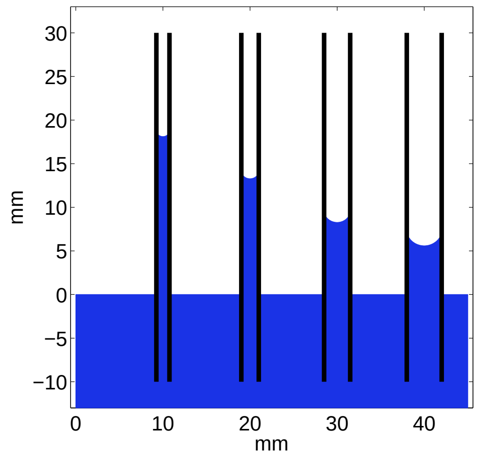
$$z_0 \simeq \frac{13.9}{R} \text{ mm}, \tag{3}$$

and the radius of curvature of the surface will be

$$c_r \simeq 1.06 \times R \text{ mm}, \tag{4}$$

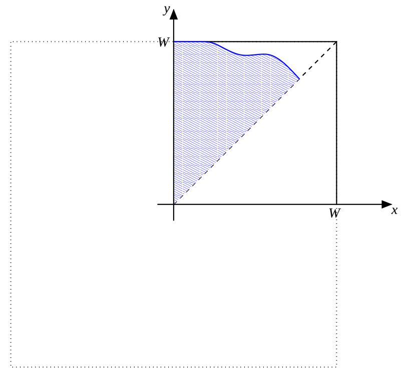
where R is measured in millimeters. So for a 2mm wide (1mm radius) tube, the water would rise 14 mm, and the radius of curvature would be 1.06 mm.

The figure below shows some typical examples



2. Inequality constraints and broken extremals: solve the isoperimetric problem inside a square region, i.e., what is the shape that contains the largest area without exceeding a given perimeter, given that the shape must be entirely contained in a square.

We will simplify the problem in a few ways. Firstly, the reflective symmetries of the problem suggest that we could consider one eight of the square, rather than the whole square (see figure).



So the inequality constraints on the problem become:

$$
\begin{aligned}
y(x) &\leq W\\
y(x) &\geq x
\end{aligned}
$$

where the left end point may move along the y-axis (between the constraints), and the right end-point is free to move along the boundary $x = y$. We will define the end-point to be (x_1, y_1).

The area of the region is easily measured by

$$A\{y\} = 8\int_0^{x_1} y - x\, dx.$$

The perimeter of the region is

$$L\{y\} = 8\int_0^{x_1} \sqrt{1 + y'^2}\, dx.$$

Ignoring the factor of 8 in each term, and including the isoperimetric constraint into the problem via a Lagrange multiplier, we obtain an objective function

$$J\{y\} = \int_0^{x_1} y - x + \lambda\sqrt{1 + y'^2}\, dx.$$

We can further simplify by noting that the problem is uninteresting for $W > R = L/2\pi$ because a circle of radius $R = L/2\pi$ satisfies the isoperimetric constraint, and fits inside the square, and by previous work this is clearly the maximal area region (though there are actually multiple possible circles that might fit). For $W < L/8$, the square can be completely filled while still satisfying the perimeter constraint (its perimeter is less than that specified), and so the square itself is the solution. The interesting cases fall in between.

Find the shape that maximizes the area without exceeding the perimeter constraint.

Solution: The Euler-Lagrange equations are

$$\frac{d}{dx}\frac{\partial f}{\partial y'} - \frac{\partial f}{\partial y} = \frac{d}{dx}\frac{\lambda y'}{\sqrt{1+y'^2}} - 1 = 0$$

This is the same as for Dido's original problem, and we know the solutions are cicular arcs. So the possible solutions to the problem involve circular arcs, and/or stretches where the inequality constraints are strict.

Consider the natural boundary conditions:

- At $x = 0$ we have x fixed, but y free, so the natural boundary condition is

$$p\big|_{x=0} = \frac{\partial f}{\partial y'}\bigg|_{x=0} = \frac{\lambda y'}{\sqrt{1+y'^2}}\bigg|_{x=0} = 0$$

  which, given $\lambda \neq 0$ means $y'|_{x=0} = 0$.
- at the right hand boundary, the natural boundary condition on the RHS is a transversal condition of the form

$$\left(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}\right) \cdot (-H, p) = p\frac{dy_\Gamma}{d\xi} - H\frac{dx_\Gamma}{d\xi} = 0$$

  where the vector $(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi})$ is a tangent to the curve along which the end-point must lie, i.e., $y = x$, so $(\frac{dx_\Gamma}{d\xi}, \frac{dy_\Gamma}{d\xi}) = (1, 1)$. So the transversality condition reduces to

$$\begin{aligned} p - H\big|_{x_1} &= 0 \\ \frac{\partial f}{\partial y'} - y'\frac{\partial f}{\partial y'} - f\bigg|_{x_1} &= 0 \\ \frac{\lambda(1+y')}{\sqrt{1+y'^2}} - (y-x)\bigg|_{x_1} &= 0 \end{aligned}$$

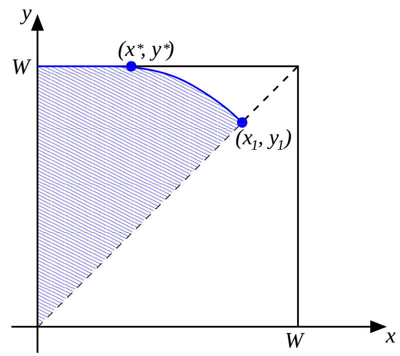
  Note though, that on the boundaryr $y = x$, so the second term vanishes, and we get

$$\frac{\lambda(1+y')}{\sqrt{1+y'^2}}\bigg|_{x_1} = 0$$

so $y' = -1$ (as above we exclude the case $\lambda = 0$ as in this case the constraint doesn't "bite"). Hence, the circular arc meets the boundary at a right angle.

A solution with zero corners looks like a circular arc, centered at the origin. We know this is optimal if the circle has perimeter L, and fits entirely inside the square, but as the perimeter grows, the largest circle to fit inside the square won't make use of the full perimeter available, and will therefore no longer be maximal. We therefore need to seek a solution with a corner.

Intuitively, the solution looks like



with a straight segment along the boundary, and then a circular arc. The "corner" or point at which the two solutions join is labelled (x^*, y^*).

At the corner, we can vary x, but not y which must be equal to W, and so the corner condition becomes

$$\begin{aligned} H\big|_{x^*-} &= H\big|_{x^*+} \\ y'\frac{\partial f}{\partial y'} - f\bigg|_{x^*-} &= y'\frac{\partial f}{\partial y'} - f\bigg|_{x^*+} \\ -W + x^* - \lambda &= \frac{\lambda y'^2}{\sqrt{1+y'^2}} - W + x^* - \lambda\sqrt{1+y'^2}\bigg|_{x^*+} \\ -\lambda &= \lambda\left[\frac{y'^2}{\sqrt{1+y'^2}} - \sqrt{1+y'^2}\right]_{x^*+} \\ -1 &= \left[\frac{-1}{\sqrt{1+y'^2}}\right]_{x^*+} \\ -1\ \sqrt{1+y'^2}\bigg|_{x^*+} &= 1 \\ y'\big|_{x^*+} &= 0 \end{aligned}$$

where $y' = 0$ on the LHS of x^*, and $y = W$ and $x = x^*$ on both LHS and RHS. Notice that the condition ensures that the curve is both continuous, and has a continuous derivative.

The perimeter of such a solution with a circular arc that has radius $r = W - x^*$ is

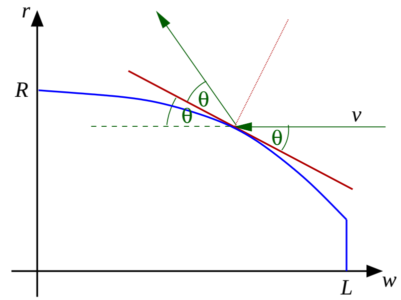
$$L = 8x^* + 2\pi r = [8 - 2\pi]x^* + 2\pi W,$$

because the 8 circular arcs taken together form a full circle. Equating the perimeter to fixed L we get

$$x^* = \frac{L - 2\pi W}{8 - 2\pi}.$$

which gives a positive value in $[0, W]$ given the perimeter lies in the bounds given in the problem.

3 Terminal costs and optimal control: We originally posed Newton's aerodynamic nose-cone problem for a nose cone pointed upwards (with flow downwards). We could equally have posed it with the flow from right to left as in the figure below, where the shape is described by $r(u)$, u being the horizontal axis.



In this case, a similar simplification of the problem reduces the function of interest to

$$\frac{1}{2\pi} F\{r\} = \frac{1}{2} r(L)^2 + \int_0^L \frac{r r'^3}{1 + r'^2} \, dw$$

Questions:

- Show that the above function results from a simple transformation of the previous problem.
- Use natural end-point conditions for a problem with a terminal cost to determine an equation to find $r(L)$.

Solutions: The form of the functional comes from the fact that (similarly to before)

$$r' = -\tan\theta,$$

and adopting the same scaling as before $2vm = 1$ the drag at a point is proportional to $\sin^2\theta$, so the total drag at u is proportional to $2\pi r \sin^2\theta$. Integrating, and noting that $\sin^2 = \tan^2/(1 + \tan^2) = r'^2/(1 + r'^2)$, and adding the drag of the flat nose-tip, we get the functional of interest.

As before, we can set the control variable $u = r' = \tan\theta$, and we obtain a set of DEs to solve.

However, the problem asked to show that we could obtain one problem from the other using a transformation. Take the original function $y(x)$ and note that the conditions under which we design the cone (i.e., a non-increasing function, except to the flat tip) mean that for the curved section the curve is invertable, i.e., we can write

$$x = r(w) = y^{-1}(w),$$

or $y(r(w)) = w$ so

$$\frac{dy}{dr}\frac{dr}{dw} = \frac{dy}{dx}\frac{dr}{dw} = 1$$

so

$$1/r' = y'$$

Hence our previous functional, transformed using $x = r(w)$ is

$$\begin{aligned}
\frac{1}{2\pi}F\{y\} &= \int_{r_0}^{R} \frac{x}{1+y'^2}\,dx, \\
\frac{1}{2\pi}F\{r\} &= \int_0^L \frac{r}{1+(1/r'^2)}\,\frac{dx}{dw}\,dw, \\
\frac{1}{2\pi}F\{r\} &= \int_0^L \frac{rr'^2}{r'^2+1}\,r'\,dw, \\
\frac{1}{2\pi}F\{r\} &= \int_0^L \frac{rr'^3}{1+r'^2}\,dw,
\end{aligned}$$

where r_0 is the radius of the flat tip, which is $r(L)$ in the new co-ordinates. The resistance of the flat tip is just the area πr_0^2 times $2mv = 1$, so the total functional is just

$$\frac{1}{2\pi}F\{r\} = \frac{1}{2}r(L)^2 + \int_0^L \frac{rr'^3}{1+r'^2}\,dw$$

We can equivalently write this using the constraint $u = r'$ to be

$$\frac{1}{2\pi}F\{r,u\} = \frac{1}{2}r(L)^2 + \int_0^L \frac{ru^3}{1+u^2} + \lambda(w)(u - r')\,dw$$

The end point conditions with a free end at $w = L$ and a terminal cost are

$$\left.\frac{\partial f}{\partial \dot{x}_k} + \frac{\partial \phi}{\partial x_k}\right|_{t=t_1} = 0.$$

In this case r and u are the dependent variables (equivalent of x_k) and $\phi(r,u) = r^2/2$, so the two equations are

$$\begin{aligned}
\left.\frac{\partial f}{\partial r'} + \frac{\partial \phi}{\partial r}\right|_{w=L} &= 0 \\
\left.\frac{\partial f}{\partial u'} + \frac{\partial \phi}{\partial u}\right|_{w=L} &= 0
\end{aligned}$$

Note that the second equation is an identity (we could also do a third equation in λ but that would also be an identity), so we only need the first.

$$\left.\frac{\partial f}{\partial r'} + \frac{\partial \phi}{\partial r}\right|_{w=L} = \frac{\partial}{\partial r'}\left[\frac{ru^3}{1+u^2} + \lambda(w)(u - r')\right] + r\Big|_{w=L} = -\lambda + r|_{w=L} = 0.$$

So at the end-point $w = L$, we get the value of the Lagrange multiplier is $r_0 = r(L)$.

Alternatively, without including Lagrange multipliers in the functional, we get

$$\begin{aligned}
\left.\frac{\partial f}{\partial r'} + \frac{\partial \phi}{\partial r}\right|_{w=L} &= 0 \\
\left.\frac{\partial}{\partial r'}\frac{rr'^3}{1+r'^2} + r\right|_{w=L} &= 0 \\
\left.\frac{3rr'^2(1+r'^2) - 2rr'^4}{(1+r'^2)^2} + r\right|_{w=L} &= 0 \\
\left. r\frac{r'^2(3+r'^2) + (1+r'^2)^2}{(1+r'^2)^2}\right|_{w=L} &= 0 \\
\left. r\frac{2r'^4 + 5r'^2 + 1}{(1+r'^2)^2}\right|_{w=L} &= 0
\end{aligned}$$

So either $r = 0$ or $2r'^4 + 5r'^2 + 1 = 0$.