Euler's Identity

The secret of the Universe?

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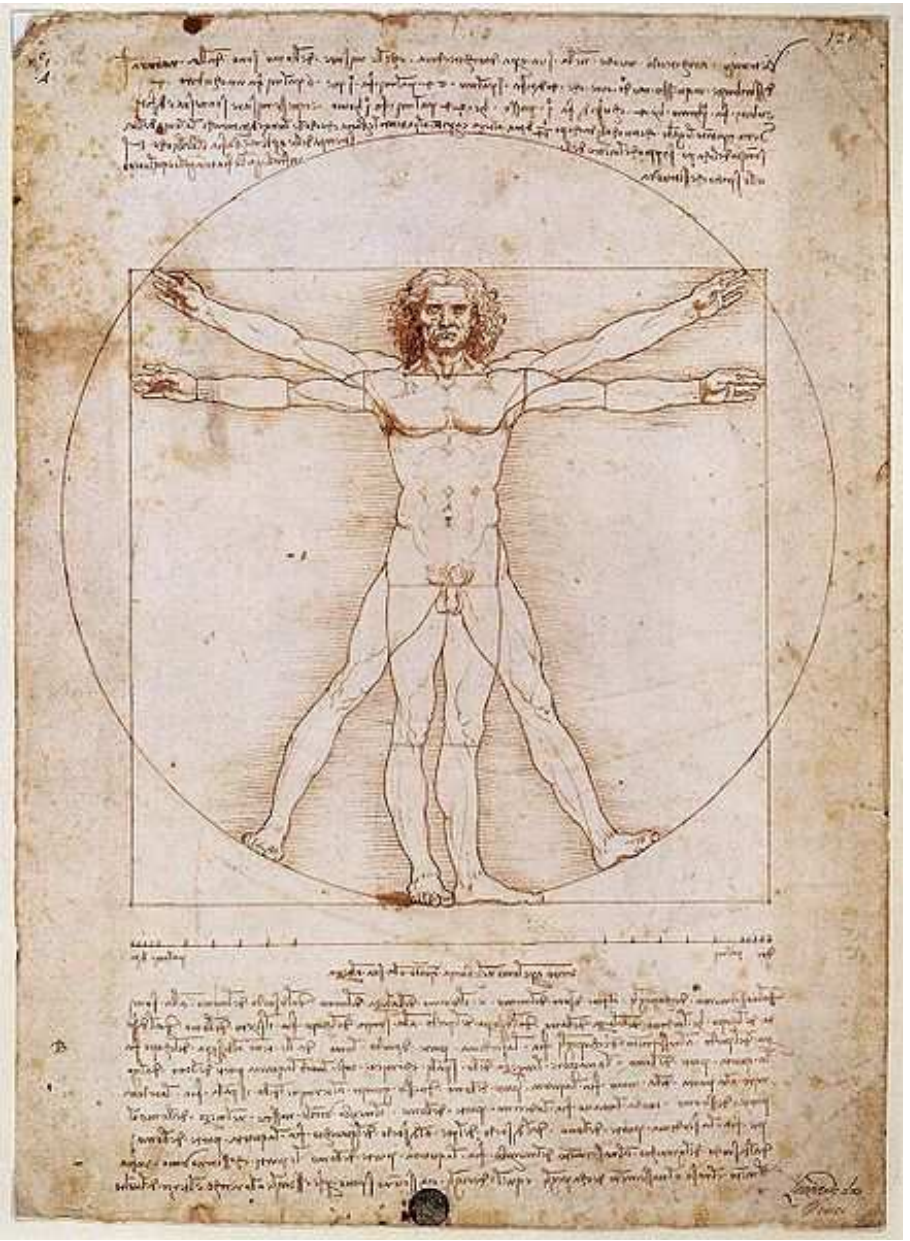
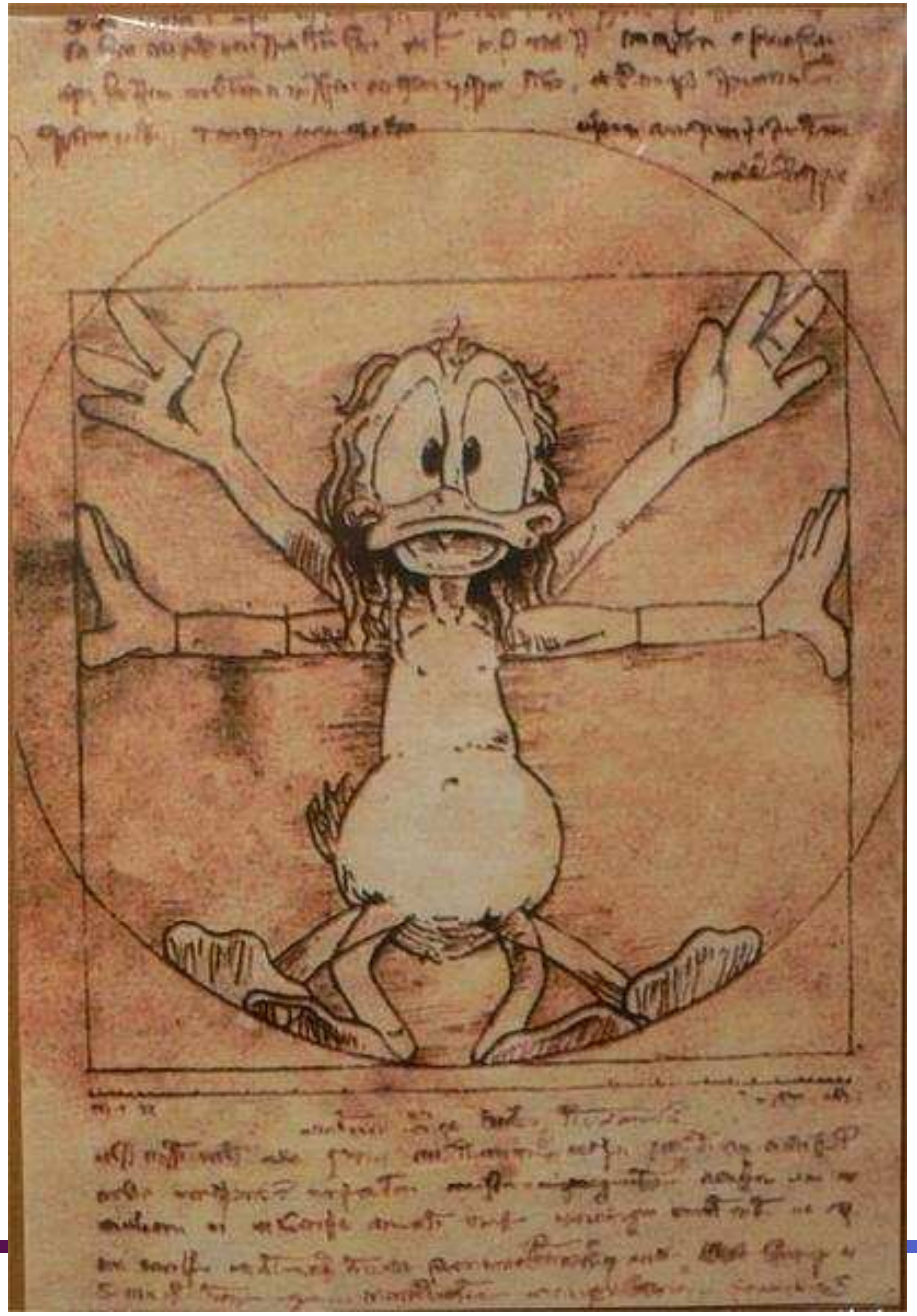
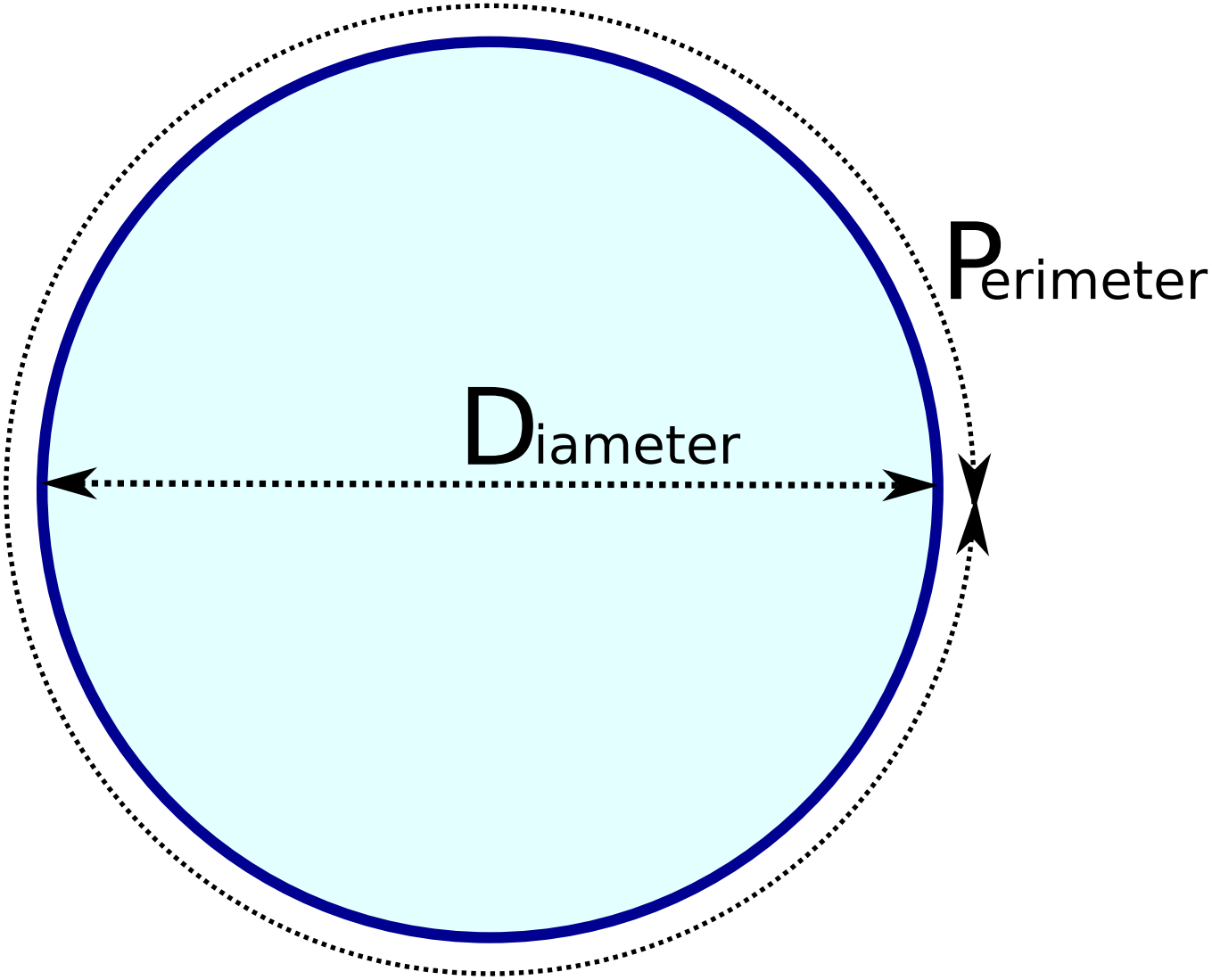
Maths as an Art



Maths as an Art



$\pi = P/D$



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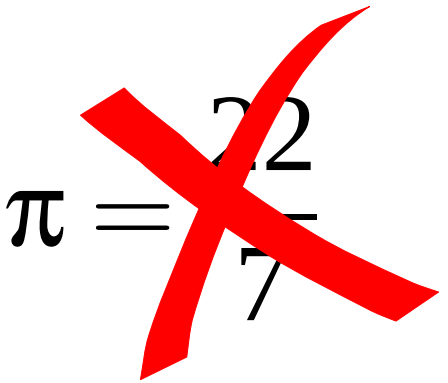
$$\pi = \frac{22}{7}$$

$\pi =$



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$$\pi = \frac{355}{113}$$

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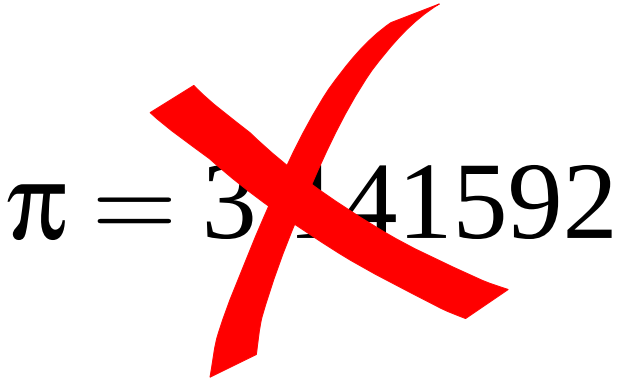
$$\pi = 3.141592$$

Mnemonic

How I wish I could recollect pi

$\pi =$



$\pi =$

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$$\pi = 3.14159265358979311599796346854 4$$

This value comes from a popular mathematical computer program called Matlab.

$\pi =$

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$$\pi = 3.14159265358979311599796346854 4$$

Matlab only gets about 15 decimal places right!

$\pi =$

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Matlab only gets about 15 decimal places right!

Can we do better?

$\pi =$

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$\pi = 3.141592653589793238462643383279502884197169399375105820974944592307816406286208998628034825342117067982148086513282306647093844609550582231725359408128481117450284102701938521105559644622948954930381964428810975665933446128475648233786783165271201909145648566923460348610454326648213393607260249141273724587006606315588174881520920962829254091715364367892590360011330530548820466521384146951941511609433057270365759591953092186117381932611793105118548074462379962749567351885752724891227938183011949129833673362440656643086021394946395224737190702179860943702770539217176293176752384674818467669405132000568127145263560827785771342757789609173637178721468440901224953430146549585371050792279689258923542019956112129021960864034418159813629774771309960518707211349999998372978049951059731732816096318595024459455346908302642522308253344685035261931188171010003137838752886587533208381420617177669147303598253490428755468731159562863882353787593751957781857780532171226806613001927876611195909216420198938095257201065485863278865936153381827968230301952035301852968995773622599413891249721775283479131515574857242454150695950829533116861727855889075098381754637464939319255060400927701671139009848824012858361603563707660104710181942955596198946767837449448255379774726847104047534646208046684259069491293313677028989152104752162056966024058038150193511253382430035587640247496473263914199272604269922796782354781636009341721641219924586315030286182974555706749838505494588586926995690927210797509302955321165344987202755960236480665499119881834797753566369807426542527862551818417574672890977772793800081647060016145249192173217214772350141441973568548161361157352552133475741849468438523323907394143334547762416862518983569485562099219222184272550254256887671790494601653466804988627232791786085784383827967976681454100953883786360950680064225125205117392984896084128488626945604241965285022210661186306744278622039194945047123713786960956364371917287467764657573962413890865832645995813390478027590099465764078951269468398352595709825822620522489407726719478268482601476990902640136394437455305068203496252451749399651431429809190659250937221696461515709858387410597885959772975498930161753928468138268683868942774155991855925245953959431049972524680845987273644695848653836736222626099124608051243884390451244136549762780797715691435997700129616089441694868555848406353422072225828488648158456028506014855825308907007589402162993954947791888134516993488890341613977039005856332293946665715408435403322439754426247610653546019158820089042763379954192312604224712383303666911291279258918247530566936123786801722736643278823537875940850461804740194781646120965421253573271516599994001643933261212606279569430277023217669427595519963761518524626842989226313746624628394394676294324903664574876050143316426881185369039960593576367924381059788594583523440038183487766676021036651294112851680251444741275897596786835228872978716495504898676529105645034054426819869546887464132849365473324789628014550566662545277393216301809082992382868249463194872592720366038663264752481199018323696147295217035628518963444792459871372859515812137805732494901842926190787327777469920699582018467444203979118525161528556245181806755987382328233456817089423175167091375383183528906417543193498302874565468683296978468538568945584179767534687316621929645775162273838253415424475246220393919563512402881801369577708696848673951627821051488616073533262350544471361658279521903603260445963177502926525329802362183422897260696780010127862254057519851395886037787374219826444953379598622524594055979635728362526510799376447179639035443250907274953423445507260014682521863346761234433241384615182399807443625707854825625395655934189973605338948323868914296482623914925977869889481627058115143153980068403948823391376964423569736375884158859162248149972501420698637256480523412977620915365157593079624145331910407557798348594914278755291519484352590493846358423104527447336506235685692366031632742193198064019068632286910967544886117316725781230130913276101577413150504868273656496174558913654305947818406719470126814314526355608276521620080003898497922606961760035295961754862758935715671300265922793025552411491927827429455385524526016189963404034307289103264243740592719433155687544574804313583266402165101251971869688367368493751247820149258266185507665641211882147393562922000988893917680314382146629937167247656627333338090898698564911871447110541278537898645329119894891406606154299686394527173114826775184220711373249912313141609782160958336342817597971577785159384032974118911379836230375964704894516698540356313749467370996053662522216565098637907060399595533620318645020712507446507255231367675498232648375091822074680344684411662275500274641855694887968882573178475954985408209124560559585003186054500499969824683389726775328610569395402414811850384390263945623376880309605624315190543259435944012591725628403794541067360689121134428149045151735306800008553232617565201906038187954318549123891346537853305514060925374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- Mathematicians have worked out π to a trillion decimal places
- The digits never end, and never repeat!

π

- π is an irrational number
  - it isn't a exact integer fraction
  - i.e., $\frac{m}{n}$ will always be an approximation
- π is a transcendental number
  - loosely, this means there isn't any finite way of representing it using integers
  - i.e., you can't "square the circle"
  - a computer can never hold the true value of π
- there is only one good, short representation of π

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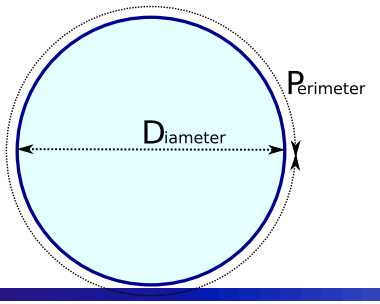
$$\pi = \frac{P}{D}$$



Euler's number e

- Euler's number is another transcendental number

$$e = 2.71828182845904523536028747135$$
$$266249775724709369995\ldots$$

- Appears in lots of places, e.g.,
  - formula for population growth
  - random processes
  - radioactive decay
- In 2004, Google announced that they would raise $2,718,281,828,

Squares and square roots

squares		square roots
$2^2 = 2 \times 2 = 4$	$\Rightarrow$	$\sqrt{4} = 2$
$3^2 = 3 \times 3 = 9$	$\Rightarrow$	$\sqrt{9} = 3$
$3.4^2 = 3.4 \times 3.4 = 11.56$	$\Rightarrow$	$\sqrt{11.56} = 3.4$
$(-1)^2 = -1 \times -1 = 1$	$\Rightarrow$	$\sqrt{-1} =???$

- When we square numbers, we always get a positive number, so we can't take the square root of a negative number!

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- When we square numbers, we always get a positive number, so we can't take the square root of a negative number!
- Or can we?

i

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- The strangest number for today is

$$i = \sqrt{-1}$$

- It isn't a real number
  - Its imaginary
    - You can't have i coconuts
    - But you can work with i like a normal number
      - you can add it
      - you can multiply it

$$3i + 4i = 7i \quad \text{and} \quad i^2 = i \times i = -1$$

Combining them

- Logically, you can operate with all of these numbers
- However, you might expect you only get a mess
- So let's take an example:

$$e^{i\pi} = ???$$

  - multiple e by itself $i\pi$ times

- Any guesses?

Euler's identity

$$e^{i\pi} = -1$$

Application

- Take a signal $f(t)$
- Then its Fourier transform is

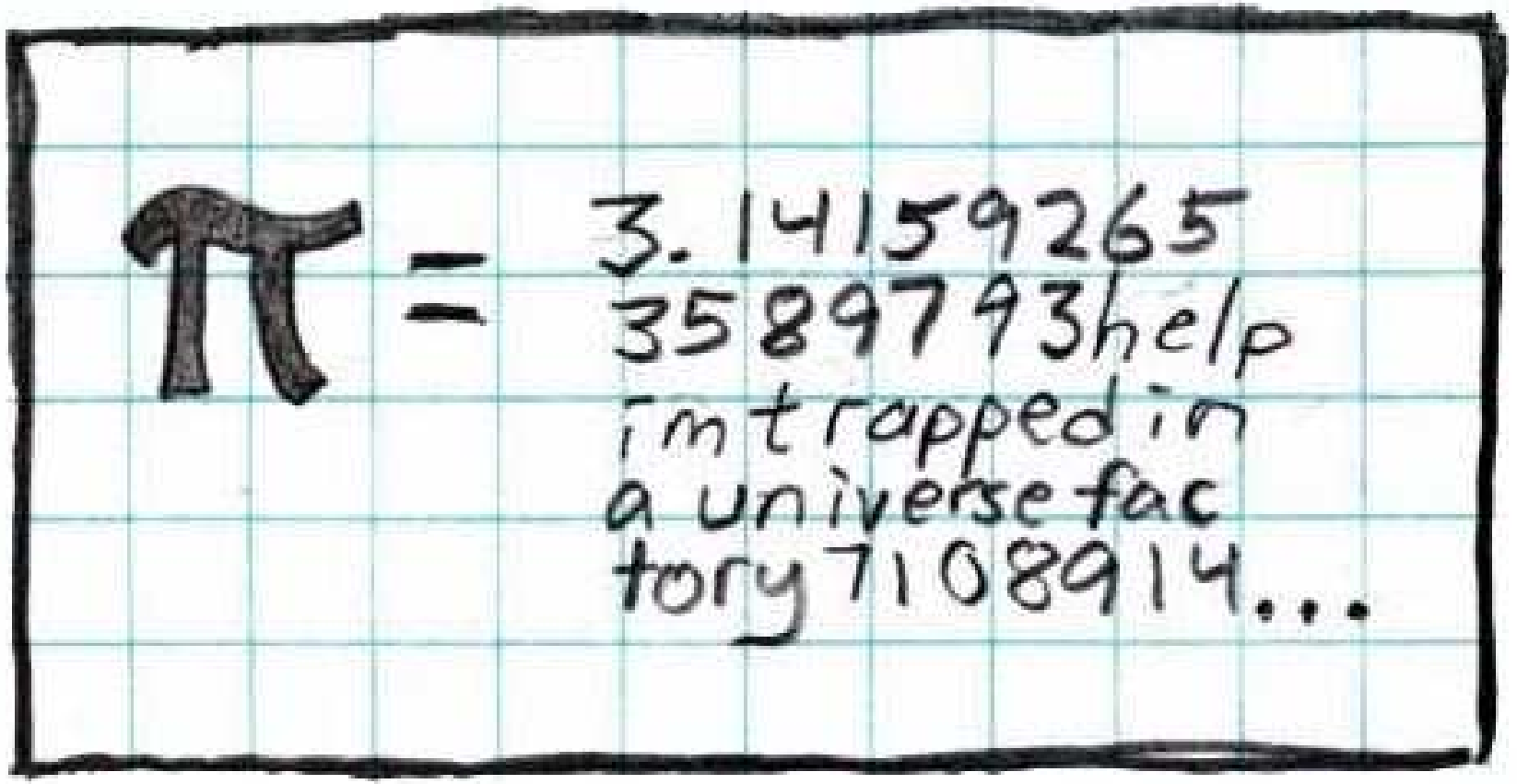
$$F(s) = \int_{-\infty}^{\infty} f(t) e^{i\pi st}\, dt$$

- The Fourier transform is one of the key parts of Signal Processing

Applications

Without Signal Processing your phone wouldn't work, or your digital camera, or your Mp3 player (to list just a few)

The End



http://xkcd.com/10/

Euler (pronounced "Oiler")

15 April 1707 –
18 September 1783

- Swiss, but lived in St. Petersburg, Russia, and in Berlin, Prussia.
- one of the most prolific mathematicians of all time
- major contributions to
  - complex analysis
  - geometry
  - calculus
  - graph theory
  - mechanics
  - fluid dynamics
  - astronomy

"Read Euler, read Euler, he is our teacher in all things,"
Pierre-Simon Laplace

Jean Baptiste Joseph Fourier

March 21, 1768 —
May 16, 1830

- son of a tailor (in Auxerre, France)
  - 12th of 15 children
- involved in the French revolution
  - at one point was arrested
- 1798 Fourier joined Napoleon's army in its invasion of Egypt as scientific adviser
  - helped in archaeological explorations.
- 1802 made Prefect of Grenoble
  - work on heat propagation, and **Fourier series**
- survived Napoleon's arrest, and return, and exile